Reynolds’ Parametricity

Patricia Johann
Appalachian State University
cs.appstate.edu/~johannp

Based on joint work with Neil Ghani, Fredrik Nordvall Forsberg, Federico Orsanigo, and Tim Revell

OPLSS 2016
Course Outline

Topic: Reynolds’ theory of parametric polymorphism for System F

Goals: - extract the fibrational essence of Reynolds’ theory
    - generalize Reynolds’ construction to very general models

• Lecture 1: Reynolds’ theory of parametricity for System F
• Lecture 2: Introduction to fibrations
• Lecture 3: A bifibrational view of parametricity
• Lecture 4: Bifibrational parametric models for System F
Course Outline

Topic: Reynolds’ theory of parametric polymorphism for System F

Goals: - extract the fibrational essence of Reynolds’ theory
    - generalize Reynolds’ construction to very general models

- Lecture 1: Reynolds’ theory of parametricity for System F
- Lecture 2: Introduction to fibrations
- Lecture 3: A bifibrational view of parametricity
- Lecture 4: Bifibrational parametric models for System F
Where Were We?

- Last time we recalled Reynolds’ standard relational parametricity
Where Were We?

- Last time we recalled Reynolds’ standard relational parametricity
- This is the main inspiration for the bifibrational model of parametricity for System F we will develop
Where Were We?

- Last time we recalled Reynolds’ standard relational parametricity
- This is the main inspiration for the bifibrational model of parametricity for System F we will develop
- View Reynolds’ construction and results through the lens of the relations (bi)fibration on Set
• Last time we recalled Reynolds’ standard relational parametricity
• This is the main inspiration for the bifibrational model of parametricity for System F we will develop
• View Reynolds’ construction and results through the lens of the relations (bi)fibration on Set
• Generalize Reynolds’ constructions to bifibrational models of System F for which we can prove (bifibrational versions of) the IEL and Abstraction Theorem
Where Were We?

- Last time we recalled Reynolds’ standard relational parametricity
- This is the main inspiration for the bifibrational model of parametricity for System F we will develop
- View Reynolds’ construction and results through the lens of the relations (bi)fibration on $\text{Set}$
- Generalize Reynolds’ constructions to bifibrational models of System F for which we can prove (bifibrational versions of) the IEL and Abstraction Theorem
- Reynolds’ construction is (ignoring size issues) such a model
Motivation: Indexed Families of Sets

- A fibration captures a family $\left( \mathcal{E}_B \right)_{B \in \mathcal{B}}$ of categories $\mathcal{E}_B$ indexed over objects of another category $\mathcal{B}$.
Motivation: Indexed Families of Sets

- A fibration captures a family \((\mathcal{E}_B)_{B \in \mathcal{B}}\) of categories \(\mathcal{E}_B\) indexed over objects of another category \(\mathcal{B}\)

- A fibration is a functor \(U : \mathcal{E} \to \mathcal{B}\)
  - \(\mathcal{B}\) is the base category of \(U\)
  - \(\mathcal{E}\) is the total category of \(U\)
Motivation: Indexed Families of Sets

- A fibration captures a family \((\mathcal{E}_B)_{B \in \mathcal{B}}\) of categories \(\mathcal{E}_B\) indexed over objects of a(whole) category \(\mathcal{B}\)

- A fibration is a functor \(U : \mathcal{E} \rightarrow \mathcal{B}\)
  - \(\mathcal{B}\) is the base category of \(U\)
  - \(\mathcal{E}\) is the total category of \(U\)

Intuitively, \(\mathcal{E} = \bigcup_{B \in \mathcal{B}} \mathcal{E}_B\)
Motivation: Indexed Families of Sets

- A fibration captures a family \((\mathcal{E}_B)_{B \in \mathcal{B}}\) of categories \(\mathcal{E}_B\) indexed over objects of another category \(\mathcal{B}\)

- A fibration is a functor \(U : \mathcal{E} \to \mathcal{B}\)
  - \(\mathcal{B}\) is the base category of \(U\)
  - \(\mathcal{E}\) is the total category of \(U\)

  Intuitively, \(\mathcal{E} = \bigcup_{B \in \mathcal{B}} \mathcal{E}_B\)

- \(U\) must have some additional properties for describing indexing
Motivation: Indexed Families of Sets

- A fibration captures a family \((\mathcal{E}_B)_{B \in \mathcal{B}}\) of categories \(\mathcal{E}_B\) indexed over objects of another category \(\mathcal{B}\).

- A fibration is a functor \(U : \mathcal{E} \to \mathcal{B}\).
  - \(\mathcal{B}\) is the base category of \(U\).
  - \(\mathcal{E}\) is the total category of \(U\).

Intuitively, \(\mathcal{E} = \bigcup_{B \in \mathcal{B}} \mathcal{E}_B\).

- \(U\) must have some additional properties for describing indexing.

- We are interested in indexing because Reynolds’ interpretations are type-indexed.
Simple case: Indexing for sets

- $\mathcal{B}$ is a set $I$ of indices,
- $\mathcal{E}$ is $X = \bigcup_{i \in I} X_i$, where $(X_i)_{i \in I}$ is a (wlog, disjoint) family of sets
- $U : X \rightarrow I$ maps each $x \in X$ to the index $i \in I$ such that $x \in X_i$
Display Maps

- Simple case: Indexing for sets
  - $\mathcal{B}$ is a set $I$ of indices,
  - $\mathcal{E}$ is $X = \bigcup_{i \in I} X_i$, where $(X_i)_{i \in I}$ is a (wlog, disjoint) family of sets
  - $U : X \to I$ maps each $x \in X$ to the index $i \in I$ such that $x \in X_i$

- $U$ is called the **display map** for $(X_i)_{i \in I}$
Display Maps

- Simple case: Indexing for sets
  - $\mathcal{B}$ is a set $I$ of indices,
  - $\mathcal{E}$ is $X = \bigcup_{i \in I} X_i$, where $(X_i)_{i \in I}$ is a (wlog, disjoint) family of sets
  - $U : X \rightarrow I$ maps each $x \in X$ to the index $i \in I$ such that $x \in X_i$

- $U$ is called the display map for $(X_i)_{i \in I}$

- It is customary to draw it vertically, like this:

$$
\begin{array}{c}
X \\
U \\
\downarrow \\
I
\end{array}
$$
Display Maps

- Simple case: Indexing for sets
  - \( \mathcal{B} \) is a set \( I \) of indices,
  - \( \mathcal{E} \) is \( X = \bigcup_{i \in I} X_i \), where \( (X_i)_{i \in I} \) is a (wlog, disjoint) family of sets
  - \( U : X \to I \) maps each \( x \in X \) to the index \( i \in I \) such that \( x \in X_i \)
- \( U \) is called the display map for \( (X_i)_{i \in I} \)
- It is customary to draw it vertically, like this:

\[
\begin{array}{c}
X \\
U \\
I
\end{array}
\]

- The set

\[
X_i = U^{-1}(i) = \{ x \in X | Ux = i \}
\]

is called the fibre of \( X \) over \( i \)
Categories from Indexed Families - Example I

- The slice category $\text{Set}/I$
Categories from Indexed Families - Example I

- The slice category $\text{Set}/I$
  - An object in $\text{Set}/I$ is a function $U : X \rightarrow I$ in $\text{Set}$
Categories from Indexed Families - Example I

- The slice category $\text{Set}/I$
  - An object in $\text{Set}/I$ is a function $U : X \to I$ in $\text{Set}$
  - A morphism from $U' : X' \to I$ and $U : X \to I$ in $\text{Set}/I$ is a function $g : X' \to X$ in $\text{Set}$ such that $U \circ g = U'$

![Diagram](attachment:image.png)
Categories from Indexed Families - Example I

- The slice category $\text{Set}/I$
  - An object in $\text{Set}/I$ is a function $U : X \to I$ in $\text{Set}$
  - A morphism from $U' : X' \to I$ and $U : X \to I$ in $\text{Set}/I$ is a function $g : X' \to X$ in $\text{Set}$ such that $U \circ g = U'$

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{U'} & & \downarrow{U} \\
I & \xleftarrow{U} & I
\end{array}
\]

- We can view $g$ as a family of functions $(g_i)_{i \in I}$, where $g_i : X'_i \to X_i$
Categories from Indexed Families - Example I

- The **slice category** $\text{Set}/I$
  - An **object** in $\text{Set}/I$ is a function $U : X \to I$ in $\text{Set}$
  - A **morphism** from $U' : X' \to I$ and $U : X \to I$ in $\text{Set}/I$ is a function $g : X' \to X$ in $\text{Set}$ such that $U \circ g = U'$

$$
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{U'} & & \downarrow{U} \\
I & \xleftarrow{U} & I
\end{array}
$$

- We can view $g$ as a family of functions $(g_i)_{i \in I}$, where $g_i : X'_i \to X_i$
- Identities and composition are inherited from $\text{Set}$
Categories from Indexed Families - Example II

- The arrow category $\text{Set}^{\to}$
Categories from Indexed Families - Example II

- The arrow category $\text{Set}^\rightarrow$
  - An object of $\text{Set}^\rightarrow$ is a function $U : X \to I$ in $\text{Set}$ for some index set $I$
The arrow category $\text{Set}^\to$

- An object of $\text{Set}^\to$ is a function $U : X \to I$ in $\text{Set}$ for some index set $I$.

- A morphism from $U' : Y \to J$ to $U : X \to I$ in $\text{Set}^\to$ is a pair $(g : Y \to X, f : J \to I)$ of functions in $\text{Set}$ such that $U \circ g = f \circ U'$.

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
U' \downarrow & & \downarrow U \\
J & \xrightarrow{f} & I
\end{array}
\]
Categories from Indexed Families - Example II

- The arrow category $\mathbf{Set}^\to$
  - An object of $\mathbf{Set}^\to$ is a function $U : X \to I$ in $\mathbf{Set}$ for some index set $I$
  - A morphism from $U' : Y \to J$ to $U : X \to I$ in $\mathbf{Set}^\to$ is a pair $(g : Y \to X, f : J \to I)$ of functions in $\mathbf{Set}$ such that $U \circ g = f \circ U'$

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow{U'} & & \downarrow{U} \\
J & \xrightarrow{f} & I
\end{array}
\]

- We can view $g$ as a family of functions $(g_j)_{j \in J}$, where $g_j : Y_j \to X_{f(j)}$ (since $g(y) \in U^{-1}(f(j))$ for any $y \in Y_j = U'^{-1}(j)$ )
Categories from Indexed Families - Example II

- The arrow category $\mathbf{Set}^\rightarrow$
  - An object of $\mathbf{Set}^\rightarrow$ is a function $U : X \to I$ in $\mathbf{Set}$ for some index set $I$
  - A morphism from $U' : Y \to J$ to $U : X \to I$ in $\mathbf{Set}^\rightarrow$ is a pair $(g : Y \to X, f : J \to I)$ of functions in $\mathbf{Set}$ such that $U \circ g = f \circ U'$

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow_{U'} & & \downarrow_{U} \\
J & \xrightarrow{f} & I
\end{array}
\]

- We can view $g$ as a family of functions $(g_j)_{j \in J}$, where $g_j : Y_j \to X_{f(j)}$ (since $g(y) \in U^{-1}(f(j))$ for any $y \in Y_j = U'^{-1}(j)$)

- Identities and composition are componentwise inherited from $\mathbf{Set}$. 

Categories from Indexed Families - Example II

- The arrow category $\text{Set}^{\to}$
  - An object of $\text{Set}^{\to}$ is a function $U : X \to I$ in $\text{Set}$ for some index set $I$
  - A morphism from $U' : Y \to J$ to $U : X \to I$ in $\text{Set}^{\to}$ is a pair $(g : Y \to X, f : J \to I)$ of functions in $\text{Set}$ such that $U \circ g = f \circ U'$

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow{U'} & & \downarrow{U} \\
J & \xrightarrow{f} & I
\end{array}
$$

- We can view $g$ as a family of functions $(g_j)_{j \in J}$, where $g_j : Y_j \to X_{f(j)}$ (since $g(y) \in U^{-1}(f(j))$ for any $y \in Y_j = U'^{-1}(j)$)

- Identities and composition are componentwise inherited from $\text{Set}$.

- $\text{Set}^{\to}$ induces a codomain functor $\text{cod} : \text{Set}^{\to} \to \text{Set}$ mapping $U : X \to I$ to $I$ and $(g, f)$ to $f$
Substitution

- Consider $U : X \rightarrow I$ for $X = (X_i)_{i \in I}$ for some index set $I$
Substitution

- Consider $U : X \to I$ for $X = (X_i)_{i \in I}$ for some index set $I$.
- **Substitution** along $f : J \to I$ turns the family $(X_i)_{i \in I}$ into a family $(Y_j)_{j \in J}$ such that $Y_j = X_{f(j)}$. 
Substitution

- Consider $U : X \to I$ for $X = (X_i)_{i \in I}$ for some index set $I$

- **Substitution** along $f : J \to I$ turns the family $(X_i)_{i \in I}$ into a family $(Y_j)_{j \in J}$ such that $Y_j = X_{f(j)}$

- $(Y_j)_{j \in J}$ is obtained by pullback of $U$ along $f$

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
U' \downarrow & & \downarrow U \\
J & \xrightarrow{f} & I
\end{array}
$$
Substitution

- Consider $U : X \rightarrow I$ for $X = (X_i)_{i \in I}$ for some index set $I$

- **Substitution** along $f : J \rightarrow I$ turns the family $(X_i)_{i \in I}$ into a family $(Y_j)_{j \in J}$ such that $Y_j = X_{f(j)}$

- $(Y_j)_{j \in J}$ is obtained by pullback of $U$ along $f$

  $\begin{array}{c}
  Y \xrightarrow{g} X \\
  \downarrow U' \\
  J \xrightarrow{f} I
  \end{array}$

- $Y = \{(j, x) \in J \times X \mid U(x) = f(j)\}$ with projection functions $g$ and $U'$
Substitution

- Consider $U : X \to I$ for $X = (X_i)_{i \in I}$ for some index set $I$
- **Substitution** along $f : J \to I$ turns the family $(X_i)_{i \in I}$ into a family $(Y_j)_{j \in J}$ such that $Y_j = X_{f(j)}$
- $(Y_j)_{j \in J}$ is obtained by pullback of $U$ along $f$

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow{U'} & & \downarrow{U} \\
J & \xrightarrow{f} & I
\end{array}
$$

- $Y = \{(j, x) \in J \times X \mid U(x) = f(j)\}$ with projection functions $g$ and $U'$
- $U' : Y \to J$ gives a new family of sets $(Y_j)_{j \in J}$ whose fibres are
  
  $Y_j = U'^{-1}(j) = \{x \in X \mid U(x) = f(j)\} = U^{-1}(f(j)) = X_{f(j)}$
Substitution

- Consider $U : X \to I$ for $X = (X_i)_{i \in I}$ for some index set $I$
- **Substitution** along $f : J \to I$ turns the family $(X_i)_{i \in I}$ into a family $(Y_j)_{j \in J}$ such that $Y_j = X_{f(j)}$
- $(Y_j)_{j \in J}$ is obtained by pullback of $U$ along $f$

$$
\begin{array}{c}
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
J & \xrightarrow{f} & I
\end{array}
\end{array}
$$

- $Y = \{(j, x) \in J \times X \mid U(x) = f(j)\}$ with projection functions $g$ and $U'$
- $U' : Y \to J$ gives a new family of sets $(Y_j)_{j \in J}$ whose fibres are
  $$
  Y_j = U'^{-1}(j) = \{x \in X \mid U(x) = f(j)\} = U^{-1}(f(j)) = X_{f(j)}
  $$
- We usually write $f^*(U)$ for the display map $U'$
Substitution - Example 1

- Let \( f \) be an element \( f : \{\ast\} \rightarrow I \)
Substitution - Example 1

- Let $f$ be an element $f : \{\ast\} \rightarrow I$
- Then $f$ picks out an element $i$ of $I$ (i.e., $f(\ast) = i$)
**Substitution - Example 1**

- Let $f$ be an element $f : \{\ast\} \to I$
- Then $f$ picks out an element $i$ of $I$ (i.e., $f(\ast) = i$)
- $Y_* = U'^{-1}(\ast) = \{x \in X \mid U(x) = i\} = U^{-1}(i) = X_i$
Let $f$ be an element $f : \{\ast\} \rightarrow I$

Then $f$ picks out an element $i$ of $I$ (i.e., $f(\ast) = i$)

$Y_\ast = U'_{-1}(\ast) = \{x \in X \mid U(x) = i\} = U^{-1}(i) = X_i$

Thus $Y = \bigcup_{j \in \{\ast\}} Y_j = Y_\ast = X_i$
Substitution - Example 1

- Let $f$ be an element $f : \{\ast\} \to I$
- Then $f$ picks out an element $i$ of $I$ (i.e., $f(\ast) = i$)
- $Y_\ast = U'^{-1}(\ast) = \{x \in X \mid U(x) = i\} = U^{-1}(i) = X_i$
- Thus $Y = \bigcup_{j \in \{\ast\}} Y_j = Y_\ast = X_i$
- So substituting along a particular element of $I$ selects the fibre of $X$ over that element
Substitution - Example 2

- Let $f$ be a non-indexed set $f : J \rightarrow \{\ast\}$
Substitution - Example 2

- Let $f$ be a non-indexed set $f : J \rightarrow \{\ast\}$
- Then, for every $j \in J$,

$$Y_j = U^{-1}(j) = \{x \in X \mid U(x) = \ast\} = U^{-1}(\ast) = X_\ast = X$$
Substitution - Example 2

- Let $f$ be a non-indexed set $f : J \rightarrow \{\ast\}$

- Then, for every $j \in J$,
  \[
  Y_j = U'^{-1}(j) = \{x \in X \mid U(x) = \ast\} = U^{-1}(\ast) = X = X
  \]

- So $Y = \bigcup_{j \in J} Y_j = J \times X$ (since the $Y_j$ are disjoint)
Substitution - Example 3

- Let $f$ be a projection $f : I \times J \to I$
Substitution - Example 3

- Let $f$ be a projection $f : I \times J \rightarrow I$
- Then, for every pair $(i, j)$,

$$Y_{(i,j)} = U'^{-1}(i, j) = \{x \in X | U(x) = i\} = U^{-1}(i) = X_i$$
Substitution - Example 3

- Let $f$ be a projection $f : I \times J \to I$

- Then, for every pair $(i, j)$,
  \[
  Y_{(i,j)} = U'(i, j) = \{ x \in X | U(x) = i \} = U^{-1}(i) = X_i
  \]

- So $Y = \bigcup_{(i,j) \in I \times J} Y_{(i,j)} = \bigcup_{(i,j) \in I \times J} X_i = X_i \times J$
Substitution - Example 3

- Let $f$ be a projection $f : I \times J \rightarrow I$
- Then, for every pair $(i, j)$,

$$Y_{(i,j)} = U'^{-1}(i, j) = \{x \in X \mid U(x) = i\} = U^{-1}(i) = X_i$$
- So $Y = \bigcup_{(i,j) \in I \times J} Y_{(i,j)} = \bigcup_{(i,j) \in I \times J} X_i = X_i \times J$
- There is a “dummy” index $j$ in the family $f^*(U)$ that plays no role
Substitution - Example 3

• Let $f$ be a projection $f : I \times J \rightarrow I$

• Then, for every pair $(i, j)$,

$$Y_{(i,j)} = U'^{-1}(i, j) = \{x \in X | U(x) = i\} = U^{-1}(i) = X_i$$

• So $Y = \bigcup_{(i,j) \in I \times J} Y_{(i,j)} = \bigcup_{(i,j) \in I \times J} X_i = X_i \times J$

• There is a “dummy” index $j$ in the family $f^*(U)$ that plays no role

• Logically speaking, substitution along a projection is weakening
Substitution - Example 4

- Let $f$ be a diagonal map $f : I \to I \times I$
Substitution - Example 4

- Let $f$ be a diagonal map $f : I \to I \times I$
- Then, for every $i \in I$,
  \[
  Y_i = U'^{-1}(i) = \{ x \in X \mid U(x) = (i, i) \} = U^{-1}(i, i) = X_{(i,i)}
  \]
Substitution - Example 4

• Let \( f \) be a diagonal map \( f : I \to I \times I \)

• Then, for every \( i \in I \),

\[
Y_i = U^{-1}(i) = \{ x \in X \mid U(x) = (i, i) \} = U^{-1}(i, i) = X_{(i, i)}
\]

• So \( Y = \bigcup_{i \in I} Y_i = \bigcup_{(i, i) \in I \times I} X_{(i, i)} \)
Let $f$ be a diagonal map $f: I \to I \times I$

Then, for every $i \in I$,

$$Y_i = U'^{-1}(i) = \{x \in X \mid U(x) = (i,i)\} = U^{-1}(i,i) = X_{(i,i)}$$

So $Y = \bigcup_{i \in I} Y_i = \bigcup_{(i,i) \in I \times I} X_{(i,i)}$

In other words, $Y$ is restriction of $\bigcup_{(i,i') \in I \times I} X_{(i,i')}$ to the diagonal $i = i'$
Substitution - Example 4

• Let \( f \) be a diagonal map \( f : I \rightarrow I \times I \)

• Then, for every \( i \in I \),

\[
Y_i = U^{-1}(i) = \{x \in X | U(x) = (i, i)\} = U^{-1}(i, i) = X_{(i,i)}
\]

• So \( Y = \bigcup_{i \in I} Y_i = \bigcup_{(i,i) \in I \times I} X_{(i,i)} \)

• In other words, \( Y \) is restriction of \( \bigcup_{(i,i') \in I \times I} X_{(i,i')} \) to the diagonal \( i = i' \)

• Logically speaking, substitution along a diagonal is contraction
The pair \((g, f)\) in the pullback diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
J & \xrightarrow{f} & I
\end{array}
\]

\[f^*(U) \xrightarrow{J} U \]

is a morphism from \(f^*(U)\) to \(U\) in the arrow category \(\text{Set}^\to\).
The pair \((g, f)\) in the pullback diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow^{f^*(U)} & & \downarrow^{U} \\
J & \xrightarrow{f} & I
\end{array}
\]

is a morphism from \(f^*(U)\) to \(U\) in the arrow category \(\textbf{Set}^{\rightarrow}\).

We call \((g, f)\) a **substitution morphism** from \(f^*(U)\) to \(U\).
(g, f) is such that if
- $U'' : Z \to K$ is any object in $\text{Set}^\to$
- $(g', f') : U'' \to U$ is a morphism in $\text{Set}^\to$
- $f' : K \to I$ factors through $f : J \to I$ via $v : K \to J$ (i.e., $f' = f \circ v$)
(g, f) is such that if
- $U'' : Z \to K$ is any object in $\text{Set}^\to$
- $(g', f') : U'' \to U$ is a morphism in $\text{Set}^\to$
- $f' : K \to I$ factors through $f : J \to I$ via $v : K \to J$ (i.e., $f' = f \circ v$)
then there exists a unique $h : Z \to Y$ in $\text{Set}^\to$ such that
- $\text{cod}(h, v) = v$ for $\text{cod} : \text{Set}^\to \to \text{Set}$
- $g \circ h = g'$
• (g, f) is such that if
  – U'' : Z → K is any object in Set→
  – (g', f') : U'' → U is a morphism in Set→
  – f' : K → I factors through f : J → I via v : K → J (i.e., f' = f ∘ v)
then there exists a unique h : Z → Y in Set→ such that
  – cod(h, v) = v for cod : Set→ → Set
  – g ∘ h = g'

\[
\begin{array}{ccc}
Z & \overset{g'}{\longrightarrow} & Y \\
\downarrow h & & \downarrow g \\
U'' & \overset{f^*(U)}{\longrightarrow} & X \\
K & \overset{v}{\longrightarrow} & J \\
\downarrow f' & & \downarrow f \\
& \overset{f'}{\longrightarrow} & I \\
& & \downarrow U
\end{array}
\]

• That is, (g, f) is the best substitution morphism from f*(U) to U
• \((g, f)\) is such that if
  - \(U'' : Z \to K\) is any object in \(\text{Set}^\to\)
  - \((g', f') : U'' \to U\) is a morphism in \(\text{Set}^\to\)
  - \(f' : K \to I\) factors through \(f : J \to I\) via \(v : K \to J\) (i.e., \(f' = f \circ v\))
then there exists a unique \(h : Z \to Y\) in \(\text{Set}^\to\) such that
  - \(\text{cod}(h, v) = v\) for \(\text{cod} : \text{Set}^\to \to \text{Set}\)
  - \(g \circ h = g'\)

That is, \((g, f)\) is the best substitution morphism from \(f^*(U)\) to \(U\)

• The existence of such best substitution morphisms is what makes \(\text{cod} : \text{Set}^\to \to \text{Set}\) a fibration
Cartesian Morphisms

Let $U : \mathcal{E} \to \mathcal{B}$ be a functor
**Cartesian Morphisms**

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor
- A morphism $g : Q \to P$ in $\mathcal{E}$ is cartesian over $f : X \to Y$ in $\mathcal{B}$ if
Cartesian Morphisms

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor.

- A morphism $g : Q \to P$ in $\mathcal{E}$ is \textit{cartesian} over $f : X \to Y$ in $\mathcal{B}$ if
  \begin{itemize}
  \item $Ug = f$
  \end{itemize}
Cartesian Morphisms

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor
- A morphism $g : Q \to P$ in $\mathcal{E}$ is cartesian over $f : X \to Y$ in $\mathcal{B}$ if
  - $Ug = f$
  - for every $g' : Q' \to P$ in $\mathcal{E}$ with $Ug' = f \circ v$ for some $v : UQ' \to X$,
    there exists a unique $h : Q' \to Q$ with $Uh = v$ and $g' = g \circ h$
**Cartesian Morphisms**

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor

- A morphism $g : Q \to P$ in $\mathcal{E}$ is cartesian over $f : X \to Y$ in $\mathcal{B}$ if
  - $Ug = f$
  - for every $g' : Q' \to P$ in $\mathcal{E}$ with $Ug' = f \circ v$ for some $v : UQ' \to X$, there exists a unique $h : Q' \to Q$ with $Uh = v$ and $g' = g \circ h$
Let $U : \mathcal{E} \to \mathcal{B}$ be a functor.
Opcartesian Morphisms

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor
- A morphism $g : P \to Q$ in $\mathcal{E}$ is opcartesian over $f : X \to Y$ in $\mathcal{B}$ if
Opcartesian Morphisms

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor
- A morphism $g : P \to Q$ in $\mathcal{E}$ is opcartesian over $f : X \to Y$ in $\mathcal{B}$ if
  - $Ug = f$
Opcartesian Morphisms

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor
- A morphism $g : P \to Q$ in $\mathcal{E}$ is opcartesian over $f : X \to Y$ in $\mathcal{B}$ if
  - $Ug = f$
  - for every $g' : P \to Q'$ in $\mathcal{E}$ with $Ug' = v \circ f$ for some $v : Y \to UQ'$, there exists a unique $h : Q \to Q'$ with $Uh = v$ and $g' = h \circ g$
Opcartesian Morphisms

- Let $U : \mathcal{E} \to \mathcal{B}$ be a functor
- A morphism $g : P \to Q$ in $\mathcal{E}$ is opcartesian over $f : X \to Y$ in $\mathcal{B}$ if
  - $Ug = f$
  - for every $g' : P \to Q'$ in $\mathcal{E}$ with $Ug' = v \circ f$ for some $v : Y \to UQ'$, there exists a unique $h : Q \to Q'$ with $Uh = v$ and $g' = h \circ g$
Observations and Notation

- Let $P$ in $\mathcal{E}$ and $f : X \rightarrow Y$ with $UP = Y$
Observations and Notation

- Let $P$ in $\mathcal{E}$ and $f : X \to Y$ with $UP = Y$
- (Op)cartesian morphisms over $f$ wrt $P$ are unique up to isomorphism
Observations and Notation

- Let $P$ in $\mathcal{E}$ and $f : X \to Y$ with $UP = Y$
- (Op)cartesian morphisms over $f$ wrt $P$ are unique up to isomorphism
- $f_P$ is the cartesian morphism over $f$ with codomain $P$
Observations and Notation

- Let $P$ in $\mathcal{E}$ and $f : X \to Y$ with $UP = Y$
- (Op)cartesian morphisms over $f$ wrt $P$ are unique up to isomorphism
- $f_P^§$ is the cartesian morphism over $f$ with codomain $P$
- $f_P^P$ is the opcartesian morphism over $f$ with domain $P$
Observations and Notation

- Let $P$ in $\mathcal{E}$ and $f : X \rightarrow Y$ with $UP = Y$
- (Op)cartesian morphisms over $f$ wrt $P$ are unique up to isomorphism
- $f_P^\$ is the cartesian morphism over $f$ with codomain $P$
- $f_P^P$ is the opcartesian morphism over $f$ with domain $P$
- $f^*P$ is the domain of $f_P^\$

Observations and Notation

- Let $P$ in $\mathcal{E}$ and $f : X \to Y$ with $UP = Y$
- (Op)cartesian morphisms over $f$ wrt $P$ are unique up to isomorphism
- $f_P^\$ is the cartesian morphism over $f$ with codomain $P$
- $f_P^P$ is the opcartesian morphism over $f$ with domain $P$
- $f^*P$ is the domain of $f_P^\$
- $\Sigma_f P$ is the codomain of $f_P^P$
**Fibrations and Opfibrations**

- $U : \mathcal{E} \to \mathcal{B}$ is a **fibration** if for every object $P$ of $\mathcal{E}$ and every $f : X \to UP$ in $\mathcal{B}$, there is a cartesian morphism $f^*_P : Q \to P$ in $\mathcal{E}$ over $f$
Fibrations and Opfibrations

• $U : \mathcal{E} \to \mathcal{B}$ is a fibration if for every object $P$ of $\mathcal{E}$ and every $f : X \to UP$ in $\mathcal{B}$, there is a cartesian morphism $f^\#_P : Q \to P$ in $\mathcal{E}$ over $f$

• $U : \mathcal{E} \to \mathcal{B}$ is an opfibration if for every object $P$ of $\mathcal{E}$ and every $f : UP \to Y$ in $\mathcal{B}$, there is an opcartesian morphism $f^\#_P : P \to Q$ in $\mathcal{E}$ over $f$
Fibrations and Opfibrations

• **$U : \mathcal{E} \to \mathcal{B}$** is a **fibration** if for every object $P$ of $\mathcal{E}$ and every $f : X \to UP$ in $\mathcal{B}$, there is a cartesian morphism $f^\mathcal{E}_P : Q \to P$ in $\mathcal{E}$ over $f$

• **$U : \mathcal{E} \to \mathcal{B}$** is an **opfibration** if for every object $P$ of $\mathcal{E}$ and every $f : UP \to Y$ in $\mathcal{B}$, there is an opcartesian morphism $f^\mathcal{E}_P : P \to Q$ in $\mathcal{E}$ over $f$

• **$U : \mathcal{E} \to \mathcal{B}$** is a **bifibration** if it is both a fibration and an opfibration
Fibrations and Opfibrations

- $U : \mathcal{E} \to \mathcal{B}$ is a **fibration** if for every object $P$ of $\mathcal{E}$ and every $f : X \to UP$ in $\mathcal{B}$, there is a cartesian morphism $f^\#: Q \to P$ in $\mathcal{E}$ over $f$.

- $U : \mathcal{E} \to \mathcal{B}$ is an **opfibration** if for every object $P$ of $\mathcal{E}$ and every $f : UP \to Y$ in $\mathcal{B}$, there is an opcartesian morphism $f^\#: P \to Q$ in $\mathcal{E}$ over $f$.

- $U : \mathcal{E} \to \mathcal{B}$ is a **bifibration** if it is both a fibration and an opfibration.

- If $U : \mathcal{E} \to \mathcal{B}$ is a fibration, opfibration, or bifibration, then an object $P$ in $\mathcal{E}$ is **over** its image $UP$ and similarly for morphisms.
Fibrations and Opfibrations

- **$U : \mathcal{E} \to \mathcal{B}$** is a **fibration** if for every object $P$ of $\mathcal{E}$ and every $f : X \to UP$ in $\mathcal{B}$, there is a cartesian morphism $f^\parallel_P : Q \to P$ in $\mathcal{E}$ over $f$

- **$U : \mathcal{E} \to \mathcal{B}$** is an **opfibration** if for every object $P$ of $\mathcal{E}$ and every $f : UP \to Y$ in $\mathcal{B}$, there is an opcartesian morphism $f^\parallel_P : P \to Q$ in $\mathcal{E}$ over $f$

- **$U : \mathcal{E} \to \mathcal{B}$** is a **bifibration** if it is both a fibration and an opfibration

- If $U : \mathcal{E} \to \mathcal{B}$ is a fibration, opfibration, or bifibration, then an object $P$ in $\mathcal{E}$ is **over** its image $UP$ and similarly for morphisms

- A morphism is **vertical** if it is over $id$
Fibrations and Opfibrations

- $U : \mathcal{E} \to \mathcal{B}$ is a fibration if for every object $P$ of $\mathcal{E}$ and every $f : X \to UP$ in $\mathcal{B}$, there is a cartesian morphism $f_P^\parallel : Q \to P$ in $\mathcal{E}$ over $f$

- $U : \mathcal{E} \to \mathcal{B}$ is an opfibration if for every object $P$ of $\mathcal{E}$ and every $f : UP \to Y$ in $\mathcal{B}$, there is an opcartesian morphism $f_P^\parallel : P \to Q$ in $\mathcal{E}$ over $f$

- $U : \mathcal{E} \to \mathcal{B}$ is a bifibration if it is both a fibration and an opfibration

- If $U : \mathcal{E} \to \mathcal{B}$ is a fibration, opfibration, or bifibration, then an object $P$ in $\mathcal{E}$ is over its image $UP$ and similarly for morphisms

- A morphism is vertical if it is over $id$

- The fibre $\mathcal{E}_X$ over an object $X$ in $\mathcal{B}$ is the subcategory of $\mathcal{E}$ of objects over $X$ and morphisms over $id_X$
Indexing and Reindexing Functors

- The function mapping each object $P$ of $\mathcal{E}$ to $f^*P$ extends to the reindexing functor $f^* : \mathcal{E}_Y \to \mathcal{E}_X$ along $f$ mapping each $k : P \to P'$ in $\mathcal{E}_Y$ to the (unique) morphism $f^*k$ such that $k \circ f_P = f_P' \circ f^*k$
Indexing and Reindexing Functors

- The function mapping each object $P$ of $\mathcal{E}$ to $f^*P$ extends to the reindexing functor $f^*: \mathcal{E}_Y \to \mathcal{E}_X$ along $f$ mapping each $k: P \to P'$ in $\mathcal{E}_Y$ to the (unique) morphism $f^*k$ such that $k \circ f_P^\circ = f_P'^\circ \circ f^*k$

- The function mapping each object $P$ of $\mathcal{E}$ to $\Sigma_f P$ extends to the opreindexing functor $\Sigma_f: \mathcal{E}_X \to \mathcal{E}_Y$ along $f$ mapping each $k: P \to P'$ in $\mathcal{E}_X$ to the (unique) morphism $\Sigma_f k$ such that $\Sigma_f k \circ f_P^\circ = f_P'^\circ \circ k$
New Fibrations from Old

\[ |\mathcal{C}| \text{ is the discrete category of } \mathcal{C} \]
New Fibrations from Old

- $|C|$ is the discrete category of $C$
- The discrete functor $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is induced by the restriction of $U : \mathcal{E} \to \mathcal{B}$ to $|\mathcal{E}|$
New Fibrations from Old

- $|\mathcal{C}|$ is the discrete category of $\mathcal{C}$
- The discrete functor $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is induced by the restriction of $U : \mathcal{E} \to \mathcal{B}$ to $|\mathcal{E}|$
- $\mathcal{C}^n$ is the $n$-fold product of $\mathcal{C}$ (in $\text{Cat}$)
New Fibrations from Old

- $|\mathcal{C}|$ is the discrete category of $\mathcal{C}$

- The discrete functor $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is induced by the restriction of $U : \mathcal{E} \to \mathcal{B}$ to $|\mathcal{E}|$

- $\mathcal{C}^n$ is the $n$-fold product of $\mathcal{C}$ (in $\mathbf{Cat}$)

- The $n$-fold product of $U : \mathcal{E} \to \mathcal{B}$, denoted $U^n : \mathcal{E}^n \to \mathcal{B}^n$, is given by $U^n(X_1, ..., X_n) = (UX_1, ..., UX_n)$ and $U^n(f_1, ..., f_n) = (Uf_1, ..., Uf_n)$
New Fibrations from Old

- $|\mathcal{C}|$ is the discrete category of $\mathcal{C}$
- The discrete functor $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is induced by the restriction of $U : \mathcal{E} \to \mathcal{B}$ to $|\mathcal{E}|$
- $\mathcal{C}^n$ is the $n$-fold product of $\mathcal{C}$ (in $\text{Cat}$)
- The $n$-fold product of $U : \mathcal{E} \to \mathcal{B}$, denoted $U^n : \mathcal{E}^n \to \mathcal{B}^n$, is given by $U^n(X_1, \ldots, X_n) = (UX_1, \ldots, UX_n)$ and $U^n(f_1, \ldots, f_n) = (Uf_1, \ldots, Uf_n)$
- Lemma
  1. If $U : \mathcal{E} \to \mathcal{B}$ is a functor, then $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is a bifibration, called the discrete fibration for $U$
  2. If $U$ is a (bi)fibration then so is $U^n : \mathcal{E}^n \to \mathcal{B}^n$ for any $n \in \text{Nat}$
Fibred Functors

• Let $U : \mathcal{E} \to \mathcal{B}$ and $U' : \mathcal{E}' \to \mathcal{B}'$ be fibrations
Fibred Functors

- Let $U : \mathcal{E} \to \mathcal{B}$ and $U' : \mathcal{E}' \to \mathcal{B}'$ be fibrations
- A fibred functor $F : U' \to U$ comprises two functors
  
  $$F_o : \mathcal{B}' \to \mathcal{B} \quad \text{and} \quad F_r : \mathcal{E}' \to \mathcal{E}$$

  such that
Fibred Functors

• Let $U: E \to B$ and $U': E' \to B'$ be fibrations

• A fibred functor $F: U' \to U$ comprises two functors

$$F_o: B' \to B \quad \text{and} \quad F_r: E' \to E$$

such that

- $U \circ F_r = F_o \circ U'$

$$
\begin{array}{ccc}
E' & \xrightarrow{F_r} & E \\
U' \downarrow & & \downarrow U \\
B' & \xrightarrow{F_o} & B \\
\end{array}
$$
Fibred Functors

- Let $U : \mathcal{E} \to \mathcal{B}$ and $U' : \mathcal{E}' \to \mathcal{B}'$ be fibrations
- A fibred functor $F : U' \to U$ comprises two functors $F_o : \mathcal{B}' \to \mathcal{B}$ and $F_r : \mathcal{E}' \to \mathcal{E}$ such that
  - $U \circ F_r = F_o \circ U'$
  - cartesian morphisms are preserved, i.e., if $f$ in $\mathcal{E}'$ is cartesian over $g$ in $\mathcal{B}'$ then $F_r f$ in $\mathcal{E}$ is cartesian over $F_o g$ in $\mathcal{B}$
Fibred Natural Transformations

- Let $F, F' : U' \to U$ be fibred functors
Fibred Natural Transformations

- Let $F, F' : U' \to U$ be fibred functors
- A fibred natural transformation $\eta : F' \to F$ comprises two natural transformations
  
  $\eta_o : F'_o \to F_o$ and $\eta_r : F'_r \to F_r$
Let $F, F' : U' \to U$ be fibred functors

A fibred natural transformation $\eta : F' \to F$ comprises two natural transformations

$$\eta_o : F'_o \to F_o \quad \text{and} \quad \eta_r : F'_r \to F_r$$

such that $U \circ \eta_r = \eta_o \circ U'$
Fibred Natural Transformations

- Let $F, F' : U' \to U$ be fibred functors

- A fibred natural transformation $\eta : F' \to F$ comprises two natural transformations

  $\eta_o : F'_o \to F_o \quad \text{and} \quad \eta_r : F'_r \to F_r$

such that $U \circ \eta_r = \eta_o \circ U'$
• View Reynolds’ construction and results through the lens of the relations (bi)fibration on $\text{Set}$
• View Reynolds’ construction and results through the lens of the relations (bi)fibration on \textbf{Set}

• Generalize Reynolds’ constructions to (bi)fibrational models of System F for which we can prove (fibrational versions of) the IEL and Abstraction Theorem Reynolds’ construction is (ignoring size issues) an instance
Coming Up

- View Reynolds’ construction and results through the lens of the relations (bi)fibration on $\text{Set}$
- Generalize Reynolds’ constructions to (bi)fibrational models of System F for which we can prove (fibrational versions of) the IEL and Abstraction Theorem Reynolds’ construction is (ignoring size issues) an instance
- Reynolds’ construction is (ignoring size issues) such a model
References
