Reynolds’ Parametricity

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Course Outline

Topic: Reynolds’ theory of parametric polymorphism for System F

Goals: - extract the fibrational essence of Reynolds’ theory
       - generalize Reynolds’ construction to very general models

- Lecture 1: Reynolds’ theory of parametricity for System F
- Lecture 2: Introduction to fibrations
- Lecture 3: A bifibrational view of parametricity
- Lecture 4: Bifibrational parametric models for System F
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  - terms as fibred natural transformations
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- This gives very general parametric models for System F
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  - types as fibred functors
  - terms as fibred natural transformations

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- Throughout, let Rel(U) be an equality preserving arrow fibration and ∀-fibration
Define fibred functors

\[ [\Delta \vdash \tau] : |\text{Rel}(U)|^{\Delta} \rightarrow \text{Rel}(U) \]

by
Fibrational Semantics of Types

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by

- Type variables: \([\Delta \vdash \alpha_i]_{o} \overline{X} = X_i \) and \([\Delta \vdash \alpha_i]_{r} \overline{R} = R_i \)
Fibrational Semantics of Types

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- Type variables: \([\Delta \vdash \alpha_i]_o X = X_i\) and \([\Delta \vdash \alpha_i]_r R = R_i\)
- Arrow types: \([\Delta \vdash \tau_1 \to \tau_2] = [\Delta \vdash \tau_1] \Rightarrow [\Delta \vdash \tau_2]\)
Fibrational Semantics of Types

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- Arrow types: \( \left[ \Delta \vdash \tau_1 \to \tau_2 \right] = \left[ \Delta \vdash \tau_1 \right] \Rightarrow \left[ \Delta \vdash \tau_2 \right] \)
- Forall types: \( \left[ \Delta \vdash \forall \alpha. \tau \right] = \forall \left[ \Delta, \alpha \vdash \tau \right] \)
Fibrational Semantics of Types

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- Type variables: \( [\Delta \vdash \alpha_i]_o X = X_i \) and \( [\Delta \vdash \alpha_i]_r R = R_i \)
- Arrow types: \( [\Delta \vdash \tau_1 \to \tau_2] = [\Delta \vdash \tau_1] \Rightarrow [\Delta \vdash \tau_2] \)
- Forall types: \( [\Delta \vdash \forall \alpha \cdot \tau] = \forall [\Delta, \alpha \vdash \tau] \)

- No definition for \( [\Delta \vdash \tau] \) on morphisms is needed because the domain of \( [\Delta \vdash \tau] \) is discrete
Type Interpretations are Equality Preserving

- **Proposition** The interpretation of every System F type is an equality preserving fibred functor
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- **Proof:** By induction on the structure of $\tau$

- If $\tau = \forall \alpha.\tau'$, then $[[\Delta \vdash \tau]] = \forall[[\Delta, \alpha \vdash \tau']]$ is an equality preserving fibred functor whenever $[[\Delta, \alpha \vdash \tau']]$ is, just by the definition of

$$\forall : (\|\text{Rel}(U)|^{n+1} \to_{\text{Eq}} \text{Rel}(U)) \to (\|\text{Rel}(U)|^{n} \to_{\text{Eq}} \text{Rel}(U))$$
Type Interpretations are Equality Preserving

- **Proposition** The interpretation of every System F type is an equality preserving fibred functor

- **Proof:** By induction on the structure of \( \tau \)

  - If \( \tau = \forall \alpha. \tau' \), then \([\Delta \vdash \tau] = \forall[\Delta, \alpha \vdash \tau']\) is an equality preserving fibred functor whenever \([\Delta, \alpha \vdash \tau']\) is, just by the definition of \( \forall \):

\[
\forall : (|\text{Rel}(U)|^{n+1} \rightarrow \text{Eq} \ 	ext{Rel}(U)) \rightarrow (|\text{Rel}(U)|^n \rightarrow \text{Eq} \ 	ext{Rel}(U))
\]

- Indeed, the very *existence* of \( \forall \) in a \( \forall \)-fibration requires that if \( F \) is equality preserving then so is \( \forall F \)
**Type Interpretations are Equality Preserving**

- **Proposition** The interpretation of every System F type is an equality preserving fibred functor.

- **Proof:** By induction on the structure of $\tau$.

- If $\tau = \forall \alpha. \tau'$, then $[[\Delta \vdash \tau]] = \forall [[\Delta, \alpha \vdash \tau']]$ is an equality preserving fibred functor whenever $[[\Delta, \alpha \vdash \tau']]$ is, just by the definition of

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- Indeed, the very *existence* of $\forall$ in a $\forall$-fibration requires that if $F$ is equality preserving then so is $\forall F$.

- In our model, the Identity Extension Lemma is "baked into" the interpretation of types, rather than something to be proved *post facto*. 
Type Interpretations are Equality Preserving

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- If $\tau = \forall \alpha.\tau'$, then $[\Delta \vdash \tau] = \forall [\Delta, \alpha \vdash \tau']$ is an equality preserving fibred functor whenever $[\Delta, \alpha \vdash \tau']$ is, just by the definition of
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  \]

- Indeed, the very *existence* of $\forall$ in a $\forall$-fibration requires that if $F$ is equality preserving then so is $\forall F$

- In our model, the Identity Extension Lemma is “baked into” the interpretation of types, rather than something to be proved *post facto*

- If $U$ is faithful, then the $\forall$-fibration requirement can be reformulated in terms of more basic concepts using opfibrational structure of $U$
In a CCC, for all $X$ and $Y$, there is an object $X \Rightarrow Y$ and a isomorphism
\[ \lambda : \text{Hom}(W \times X, Y) \cong \text{Hom}(W, X \Rightarrow Y) \]
that is natural in $W$.
Fibrational Semantics of Terms - The Set Up

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• The unit of this adjunction is the evaluation map

$$\text{ev}_{X,Y} = \lambda^{-1}(id_{X \Rightarrow Y}) : (X \Rightarrow Y) \times X \rightarrow Y$$
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• In a CCC, for all $X$ and $Y$, there is an object $X \Rightarrow Y$ and a isomorphism
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• The unit of this adjunction is the evaluation map
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  \]

• In a $\forall$-fibration, for every $F$ and $G$, there is are isomorphisms
  \[
  \varphi_n : \text{Hom}(F \circ \pi_n, G) \cong \text{Hom}(F, \forall_n G)
  \]
  that are natural in $n$
Fibrational Semantics of Terms - term variables

Define fibred natural transformations

$$[[\Delta; \Gamma \vdash t : \tau]] : [[\Delta \vdash \Gamma]] \to [[\Delta \vdash \tau]]$$

by
Define fibred natural transformations

\[[\Delta; \Gamma \vdash t : \tau] : [\Delta \vdash \Gamma] \rightarrow [\Delta \vdash \tau]\]

by

- If

\[
\Delta \vdash \tau_i \quad x_i : \tau_i \in \Gamma
\]

\[
\Delta; \Gamma \vdash x_i : \tau_i
\]

then

\[[\Delta; \Gamma \vdash x_i : \tau_i] = \pi_i\]
Fibrational Semantics of Terms - term variables

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- \(\pi_i\) is the \(i^{th}\) projection on both \(B\) and \(E\)
Define fibred natural transformations

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[\Delta; \Gamma \vdash t : \tau] : [\Delta \vdash \Gamma] \to [\Delta \vdash \tau]
\]

by

• If

\[
\frac{\Delta \vdash \tau_i \quad x_i : \tau_i \in \Gamma}{\Delta; \Gamma \vdash x_i : \tau_i}
\]

then

\[
[\Delta; \Gamma \vdash x_i : \tau_i] = \pi_i
\]

• $\pi_i$ is the $i^{th}$ projection on both $B$ and $E$

• This specializes to our $\text{Set}$ interpretation of variables
Fibrational Semantics of Terms - term abstractions

- If

\[
\Delta; \Gamma, x : \tau_1 \vdash t : \tau_2 \\
\Delta; \Gamma \vdash \lambda x. t : \tau_1 \rightarrow \tau_2
\]

then

\[
[\Delta; \Gamma \vdash \lambda x. t : \tau_1 \rightarrow \tau_2] : \ [\Delta \vdash] \rightarrow ([\Delta \vdash \tau_1] \Rightarrow [\Delta \vdash \tau_2])
\]

\[
[\Delta; \Gamma \vdash \lambda x. t : \tau_1 \rightarrow \tau_2] = \lambda[\Delta; \Gamma, x : \tau_1 \vdash t : \tau_2]
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Fibrational Semantics of Terms - term abstractions

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\]

\[
[\Delta; \Gamma \vdash \lambda x.t : \tau_1 \rightarrow \tau_2] = \lambda[\Delta; \Gamma, x : \tau_1 \vdash t : \tau_2]
\]

• This is sensible because

\[
[\Delta; \Gamma, x : \tau_1 \vdash t : \tau_2] : [\Delta \vdash \Gamma] \times [\Delta \vdash \tau_1] \rightarrow [\Delta \vdash \tau_2]
\]
Fibrational Semantics of Terms - term abstractions

• If

\[
\frac{\Delta; \Gamma, x : \tau_1 \vdash t : \tau_2}{\Delta; \Gamma \vdash \lambda x. t : \tau_1 \rightarrow \tau_2}
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then

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• \(\lambda\) is the right adjoint to \(\times\) in both \(\mathcal{B}\) and \(\mathcal{E}\)
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- This specializes to our \textbf{Set} interpretation of term abstractions
Fibrational Semantics of Terms - term applications

• If

\[ \Delta; \Gamma \vdash t_1 : \tau_1 \quad \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2 \]
\[ \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \]

then

\[ \llbracket \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \rrbracket \quad \llbracket \Delta \vdash \Gamma \rrbracket \to \llbracket \Delta \vdash \tau_2 \rrbracket \]
\[ \llbracket \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \rrbracket = \text{ev}_{\tau_1, \tau_2} \circ \langle \llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2 \rrbracket, \llbracket \Delta; \Gamma \vdash t_1 : \tau_1 \rrbracket \rangle \]
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\Delta; \Gamma \vdash t_1 : \tau_1 \quad \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2
\]
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\[
\Delta; \Gamma \vdash t_2 t_1 : \tau_2
\]

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\[
[\Delta; \Gamma \vdash t_2 t_1 : \tau_2] : \ [\Delta \vdash \Gamma] \to [\Delta \vdash \tau_2]
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[\Delta; \Gamma \vdash t_2 t_1 : \tau_2] = \text{ev}_{\tau_1, \tau_2} \circ \langle [\Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2], [\Delta; \Gamma \vdash t_1 : \tau_1] \rangle
\]

This is sensible because
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[\Delta; \Gamma \vdash t_1 : \tau_1] : \ [\Delta \vdash \Gamma] \to [\Delta \vdash \tau_1]
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Fibrational Semantics of Terms - term applications

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\]

\[
\Delta; \Gamma \vdash t_2 t_1 : \tau_2
\]

then

\[
\llbracket \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \to \llbracket \Delta \vdash \tau_2 \rrbracket
\]

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\]

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\]

\[
\llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \to (\llbracket \Delta \vdash \tau_1 \rrbracket \Rightarrow \llbracket \Delta \vdash \tau_2 \rrbracket)
\]
Fibrational Semantics of Terms - term applications

• If

\[
\Delta; \Gamma \vdash t_1 : \tau_1 \quad \Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2
\]

then

\[
\left[\Delta; \Gamma \vdash t_1 : \tau_1\right] : \left[\Delta \vdash \Gamma\right] \rightarrow \left[\Delta \vdash \tau_2\right]
\]

\[
\left[\Delta; \Gamma \vdash t_2 t_1 : \tau_2\right] = \text{ev}_{\tau_1, \tau_2} \circ \left<\left[\Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2\right], \left[\Delta; \Gamma \vdash t_1 : \tau_1\right]\right>
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\left[\Delta; \Gamma \vdash t_1 : \tau_1\right] : \left[\Delta \vdash \Gamma\right] \rightarrow \left[\Delta \vdash \tau_1\right]
\]

\[
\left[\Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2\right] : \left[\Delta \vdash \Gamma\right] \rightarrow (\left[\Delta \vdash \tau_1\right] \Rightarrow \left[\Delta \vdash \tau_2\right])
\]

\[
\left<\left[\Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2\right], \left[\Delta; \Gamma \vdash t_1 : \tau_1\right]\right> : \left[\Gamma\right] \rightarrow (\left[\tau_1\right] \Rightarrow \left[\tau_2\right]) \times \left[\tau_1\right]
\]
Fibrational Semantics of Terms - term applications

- If

\[ \frac{\Delta; \Gamma \vdash t_1 : \tau_1 \quad \Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2}{\Delta; \Gamma \vdash t_2 t_1 : \tau_2} \]

then

\[ \begin{array}{c}
\llbracket \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \rightarrow \llbracket \Delta \vdash \tau_2 \rrbracket \\
\llbracket \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \rrbracket = \text{ev}_{\tau_1, \tau_2} \circ \langle \llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2 \rrbracket, \llbracket \Delta; \Gamma \vdash t_1 : \tau_1 \rrbracket \rangle
\end{array} \]

- This is sensible because

\[ \begin{array}{c}
\llbracket \Delta; \Gamma \vdash t_1 : \tau_1 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \rightarrow \llbracket \Delta \vdash \tau_1 \rrbracket \\
\llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \rightarrow (\llbracket \Delta \vdash \tau_1 \rrbracket \Rightarrow \llbracket \Delta \vdash \tau_2 \rrbracket) \\
\langle \llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \rightarrow \tau_2 \rrbracket, \llbracket \Delta; \Gamma \vdash t_1 : \tau_1 \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow (\llbracket \tau_1 \rrbracket \Rightarrow \llbracket \tau_2 \rrbracket) \times \llbracket \tau_1 \rrbracket \\
\end{array} \]

- \langle - , - \rangle : (X \rightarrow Y) \times (X \rightarrow W) \rightarrow X \rightarrow (Y \times W) is \langle f, g \rangle \overline{X} = f \overline{X} \times g \overline{X}
Fibrational Semantics of Terms - term applications

• If

\[
\Delta; \Gamma \vdash t_1 : \tau_1 \quad \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2 \\
\Delta; \Gamma \vdash t_2 t_1 : \tau_2
\]

then

\[
\llbracket \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \to \llbracket \Delta \vdash \tau_2 \rrbracket \\
\llbracket \Delta; \Gamma \vdash t_2 t_1 : \tau_2 \rrbracket = \text{ev}_{\tau_1, \tau_2} \circ \langle \llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2 \rrbracket, \llbracket \Delta; \Gamma \vdash t_1 : \tau_1 \rrbracket \rangle
\]

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\llbracket \Delta; \Gamma \vdash t_1 : \tau_1 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \to \llbracket \Delta \vdash \tau_1 \rrbracket \\
\llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2 \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \to (\llbracket \Delta \vdash \tau_1 \rrbracket \Rightarrow \llbracket \Delta \vdash \tau_2 \rrbracket) \\
\langle \llbracket \Delta; \Gamma \vdash t_2 : \tau_1 \to \tau_2 \rrbracket, \llbracket \Delta; \Gamma \vdash t_1 : \tau_1 \rrbracket \rangle : \llbracket \Gamma \rrbracket \to (\llbracket \tau_1 \rrbracket \Rightarrow \llbracket \tau_2 \rrbracket) \times \llbracket \tau_1 \rrbracket
\]

• \( \langle -, - \rangle : (X \to Y) \times (X \to W) \to X \to (Y \times W) \) is \( \langle f, g \rangle X = fX \times gX \)

• This specializes to our Set interpretation of term applications
Fibrational Semantics of Terms - type abstractions

- If

\[
\begin{align*}
\Delta, \alpha; \Gamma \vdash t : \tau \\
\Delta; \Gamma \vdash \Lambda \alpha.t : \forall \alpha.\tau
\end{align*}
\]

then

\[
[\Delta; \Gamma \vdash \Lambda \alpha.t : \forall \alpha.\tau] : [\Delta \vdash \Gamma] \rightarrow [\Delta \vdash \forall \alpha.\tau] = [\Delta \vdash \Gamma] \rightarrow \forall [\Delta, \alpha \vdash \tau]
\]

\[
[\Delta; \Gamma \vdash \Lambda \alpha.t : \forall \alpha.\tau] = \phi_{\Delta}[\Delta, \alpha; \Gamma \vdash t : \tau]
\]

- This is sensible because \(\alpha\) is not free in \(\Gamma\), so

\[
[\Delta, \alpha; \Gamma \vdash t : \tau] : [\Delta, \alpha \vdash \Gamma] \rightarrow [\Delta, \alpha \vdash \tau] = [\Delta \vdash \Gamma] \circ \pi_{\Delta} \rightarrow [\Delta, \alpha \vdash \tau]
\]
Fibrational Semantics of Terms - type applications

• If

\[
\frac{\Delta; \Gamma \vdash t : \forall \alpha. \tau_2 \quad \Delta \vdash \tau_1}{\Delta; \Gamma \vdash t \tau_1 : \tau_2[\alpha \mapsto \tau_1]}
\]

then

\[
[\Delta; \Gamma \vdash t \tau_1 : \tau_2[\alpha \mapsto \tau_1]] : [\Delta \vdash \Gamma] \rightarrow [\Delta \vdash \tau_2[\alpha \mapsto \tau_1]]
\]

\[
[\Delta; \Gamma \vdash t \tau_1 : \tau_2[\alpha \mapsto \tau_1]] = \varphi_{\Delta}^{-1}[\Delta; \Gamma \vdash t : \forall \alpha. \tau_2] \circ \langle id^{\Delta}, [\Delta \vdash \tau_1] \rangle
\]

• This is sensible because

\[
[\Delta; \Gamma \vdash t : \forall \alpha. \tau_2] : [\Delta \vdash \Gamma] \rightarrow [\Delta \vdash \forall \alpha. \tau_2]
\]

\[
= [\Delta \vdash \Gamma] \rightarrow \forall [\Delta, \alpha \vdash \tau_2]
\]
Validating $\beta$- and $\eta$-Rules

- Our model is sensible by construction
Validating $\beta$- and $\eta$-Rules

- Our model is sensible by construction

- Reynolds’ model is an instance of ours, assuming a constructive metatheory — e.g., the Calculus of Constructions with impredicative Set
Validating $\beta$- and $\eta$-Rules

• Our model is sensible by construction

• Reynolds’ model is an instance of ours, assuming a constructive metatheory — e.g., the Calculus of Constructions with impredicative $\text{Set}$

• Proposition If $\Delta \vdash \tau_1$ and $\Delta, \alpha; \Gamma \vdash t : \tau_2$
  1. $\\llbracket \Delta; \Gamma \vdash (\Lambda \alpha.t) \tau_1 : \tau_2[\alpha \mapsto \tau_1] \rrbracket = \llbracket \Delta; \Gamma \vdash t[\alpha \mapsto \tau_1] : \tau_2[\alpha \mapsto \tau_1] \rrbracket$
  2. $\\llbracket \Delta; \Gamma \vdash t : \forall \beta.\tau \rrbracket = \llbracket \Delta; \Gamma \vdash \Lambda \alpha.t \alpha : \forall \beta.\tau \rrbracket$
Validating $\beta$- and $\eta$-Rules

- Our model is sensible by construction
- Reynolds’ model is an instance of ours, assuming a constructive metatheory — e.g., the Calculus of Constructions with impredicative Set

- Proposition If $\Delta \vdash \tau_1$ and $\Delta, \alpha; \Gamma \vdash t : \tau_2$
  1. $[\Delta; \Gamma \vdash (\Lambda \alpha. t) \tau_1 : \tau_2[\alpha \mapsto \tau_1]] = [\Delta; \Gamma \vdash t[\alpha \mapsto \tau_1] : \tau_2[\alpha \mapsto \tau_1]]$
  2. $[\Delta; \Gamma \vdash t : \forall \beta. \tau] = [\Delta; \Gamma \vdash \Lambda \alpha. t \alpha : \forall \beta. \tau]$

- Proposition If $\Delta; \Gamma \vdash t : \tau_1$ and $\Delta; \Gamma, x : \tau_1 \vdash t_2 : \tau_2$
  1. $[\Delta; \Gamma \vdash (\lambda x. t_2) t_1 : \tau_2] = [\Delta; \Gamma \vdash t_2[x \mapsto t_1] : \tau_2]$
  2. $[\Delta; \Gamma \vdash t : \tau_1 \rightarrow \tau_2] = [\Delta; \Gamma \vdash \lambda x. tx : \tau_1 \rightarrow \tau_2]$
Reynolds’ Abstraction Theorem, Generalized

- Our model actually gives rise to a $\lambda^2$-fibration
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- **Theorem** If \( \text{Rel}(U) \) is an equality preserving arrow fibration and a \( \forall \)-fibration, then there is a λ2-fibration in which types \( \Delta \vdash \tau \) are interpreted as equality preserving fibred functors \([\Delta \vdash \tau] : |\text{Rel}(U)|_{\Delta} \to_{\text{Eq}} \text{Rel}(U)\) and terms \( \Delta; \Gamma \vdash t : \tau \) are interpreted as fibred natural transformations \([\Delta; \Gamma \vdash t : \tau] : [\Delta \vdash \Gamma] \to [\Delta \vdash \tau] \)
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- **Theorem** If $\text{Rel}(U)$ is an equality preserving arrow fibration and a $\forall$-fibration, then there is a $\lambda^2$-fibration in which types $\Delta \vdash \tau$ are interpreted as equality preserving fibred functors $\llbracket \Delta \vdash \tau \rrbracket : \llbracket \text{Rel}(U) \rrbracket_{\Delta} \to_{\text{Eq}} \text{Rel}(U)$ and terms $\Delta; \Gamma \vdash t : \tau$ are interpreted as fibred natural transformations $\llbracket \Delta; \Gamma \vdash t : \tau \rrbracket : \llbracket \Delta \vdash \Gamma \rrbracket \to \llbracket \Delta \vdash \tau \rrbracket$

\[
\begin{array}{c}
\llbracket \text{Rel}(E) \rrbracket_{\Delta} \\
\llbracket \text{Rel}(U) \rrbracket_{\Delta} \\
\llbracket B \rrbracket_{\Delta} \times \llbracket B \rrbracket_{\Delta}
\end{array}
\xymatrix{
\llbracket \text{Rel}(E) \rrbracket_{\Delta} \ar[rr]^\llbracket \Gamma \rrbracket_r \ar[dd]_{\downarrow \llbracket t \rrbracket_r} & & \text{Rel}(E) \ar[dd]^U \\
\llbracket \text{Rel}(U) \rrbracket_{\Delta} & & \\
\llbracket B \rrbracket_{\Delta} \times \llbracket B \rrbracket_{\Delta} \ar[rrr]_{\llbracket \tau \rrbracket_0 \times \llbracket \tau \rrbracket_0} & & \llbracket B \rbracket \times \llbracket B \rbracket
}
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In particular, for every fibration $U : \mathcal{E} \to B$ whose relations fibration is an equality preserving arrow fibration and a forall fibration, for every System F type $\Delta \vdash \tau$ and term $\Delta; \Gamma \vdash t : \tau$, we get:
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In particular, for every fibration \( U : \mathcal{E} \to B \) whose relations fibration is an equality preserving arrow fibration and a forall fibration, for every System F type \( \Delta \vdash \tau \) and term \( \Delta; \Gamma \vdash t : \tau \), we get:

1. An object interpretation of \( \Delta \vdash \tau \) as a functor \( \llbracket \Delta \vdash \tau \rrbracket_o : |\mathcal{B}|^{|\Delta|} \to \mathcal{B} \)
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Additional Observations

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- These are litmus tests verifying that a model is “good”
• Parametricity entails replacing usual categorical semantics involving categories, functors, and natural transformations
Summing Up

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**Summing Up**

- Parametricity entails replacing usual categorical semantics involving categories, functors, and natural transformations with a semantics based on fibrations, fibred functors, and fibred natural transformations.

- This hits the *sweet spot* between the simplicity and “light structure” of functorial models and the ability to prove expected key results.
Examples

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- The PER model of Bainbridge et al. is also an instance (if bifibrations are understood as internal to the category of \( \omega \)-sets)
A Prescriptive General Framework

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  - changing the total category and the fibration (i.e., the functor itself) changes the notion of relational logic
- Ex: Using non-standard relations, we can construct a model of “multi-valued parametricity” over a constructively completely distributive complete non-trivial lattice of truth values
At WoLLIC’15, Neil Ghani, Fredrik Nordvall Forsberg, and Federico Orsanigo showed how to avoid baking the IEL into our framework, but rather derive it from more primitive assumptions about equality-preserving cones that can be used to interpret forall types.
Extensions

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• Clément Aubert, Fredrik Nordvall Forsberg, and I are working on extending our framework to a polymorphic calculus with computational effects (System F with effect-free constants and algebraic operations in the style of Plotkin and Power’s effectful simply-typed calculus \( \lambda_c \))
References


- Types, abstractions, and parametric polymorphism, part 2. Q. Ma and J. Reynolds. MFPS’92 [Developed the first categorical framework for parametric polymorphism (PL-categories)]

- Categorical models for Abadi and Plotkin’s logic for parametricity. L. Birkedal and R. Møgelberg. Mathematical Structures in Computer Science, 2005. [Constructs sophisticated models of parametricity and its logical structure. Also argues that not all expected consequences hold in Ma and Reynolds’ framework]

- Parametric limits. B. Dunphy and U. Reddy. LICS’04. [First model to mix fibrations with reflexive graphs, but obtains existence of initial algebras only for strictly positive functors]

- And many, many more...