

BATBAMBAMBA: Boolean and Arithmetic Languages Oregon Programming Languages Summer School

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1 BA: Boolean Arithmetic Language

Program Static Syntax

$$\begin{aligned}
 t &\in \text{TERM}, \quad n \in \mathbb{N}, \quad b \in \mathbb{B}, \quad p \in \text{PGM} = \text{TERM} \\
 t &::= \text{true} \mid \text{false} \mid \text{if } t \text{ then } t \text{ else } t \\
 &\quad \mid n \mid \text{succ}(t) \mid \text{pred}(t) \mid \text{zero?}(t) \\
 b &::= \text{true} \mid \text{false}
 \end{aligned}$$

Program Runtime Syntax

$$\begin{aligned}
 E &\in \text{ECTXT}, \quad v \in \text{VALUE}, \quad r \in \text{REDEX} \subseteq \text{PGM}, \quad f \in \text{FAULTY} \subseteq \text{PGM}, \quad \text{err} \in \text{ERROR}, \quad c \in \text{CONFIG} \quad o \in \text{OBS} \\
 v &::= b \mid n \\
 E &::= \square \mid \text{if } E \text{ then } p \text{ else } p \mid \text{succ}(E) \mid \text{pred}(E) \mid \text{zero?}(E) \\
 r &::= \text{if } v \text{ then } p \text{ else } p \mid \text{succ}(v) \mid \text{pred}(v) \mid \text{zero?}(v) \\
 f &::= \text{if } n \text{ then } p \text{ else } p \mid \text{succ}(b) \mid \text{pred}(b) \mid \text{zero?}(b) \\
 \text{err} &::= \text{mismatch} \mid \text{underflow} \\
 c &::= p \mid \text{err} \\
 o &::= v \mid \text{err}
 \end{aligned}$$
 $\rightsquigarrow \subseteq \text{REDEX} \times \text{PGM}$

Notions of Reduction

$$\begin{aligned}
 \text{if true then } p_2 \text{ else } p_3 &\rightsquigarrow p_2 \\
 \text{if false then } p_2 \text{ else } p_3 &\rightsquigarrow p_3 \\
 \text{pred}(n) &\rightsquigarrow n - 1 \quad \text{if } n > 0 \\
 \text{zero?}(0) &\rightsquigarrow \text{true} \\
 \text{zero?}(n) &\rightsquigarrow \text{false} \quad \text{if } n > 0 \\
 \text{succ}(n) &\rightsquigarrow n + 1
 \end{aligned}$$
 $\longrightarrow \subseteq \text{PGM} \times \text{CONFIG}$

Single-step Reduction

$$\begin{array}{c}
 \frac{r \rightsquigarrow p}{E[r] \longrightarrow E[p]} \\
 \hline
 E[f] \longrightarrow \text{mismatch} \\
 \hline
 E[\text{pred}(0)] \longrightarrow \text{underflow}
 \end{array}$$
 $\longrightarrow^* \subseteq \text{CONFIG} \times \text{CONFIG}$

Multi-step Reduction

$$\begin{array}{ccc}
 (\text{incl}) \frac{c_1 \longrightarrow c_2}{c_1 \longrightarrow^* c_2} & (\text{refl}) \frac{}{c \longrightarrow^* c} & (\text{trans}) \frac{c_1 \longrightarrow^* c_2 \quad c_2 \longrightarrow^* c_3}{c_1 \longrightarrow^* c_3}
 \end{array}$$
 $\text{eval}_{BA} : \text{PGM} \rightarrow \text{OBS}$

$$\begin{aligned}
 \text{eval}_{BA}(p) &= b \text{ if } p \longrightarrow^* b \\
 \text{eval}_{BA}(p) &= n \text{ if } p \longrightarrow^* n \\
 \text{eval}_{BA}(p) &= \text{mismatch} \text{ if } p \longrightarrow^* \text{mismatch} \\
 \text{eval}_{BA}(p) &= \text{underflow} \text{ if } p \longrightarrow^* \text{underflow}
 \end{aligned}$$

Safety (AKA Coherence AKA Definedness)

Conjecture 1 (Progress). For all $p \in \text{PGM}$ one of the following is true:

1. $p \in \text{VALUE}$;
2. $p \longrightarrow p'$ for some $p' \in \text{PGM}$;
3. $p \longrightarrow \text{err}$ for some $\text{err} \in \text{ERROR}$.

Conjecture 2 (Preservation (Vacuous)). If $p_1 \longrightarrow p_2$ then $p_2 \in \text{PGM}$. (uhh...?!?)

TBA: Typed Boolean Arithmetic Language

Program Static Syntax

$$t \in \text{TERM}, \quad n \in \mathbb{N}, \quad b \in \mathbb{B}, \quad \text{Same as BA}$$

$$T \in \text{TYPE}, \quad p \in \text{PGM} = \{t \in \text{TERM} \mid \exists T \in \text{TYPE}. \vdash t : T\}$$

$$T ::= \text{Nat} \mid \text{Bool}$$

Program Runtime Syntax

$$E \in \text{ECTXT}, \quad v \in \text{VALUE}, \quad r \in \text{REDEX} \subseteq \text{PGM}, \quad \text{Same grammar as BA (over updated p)}$$

$$\text{err} \in \text{ERROR}, \quad c \in \text{CONFIG} \quad o \in \text{OBS}$$

$$v ::= b \mid n$$

$$r ::= \text{if } v \text{ then } p \text{ else } p \mid \text{succ}(v) \mid \text{pred}(v) \mid \text{zero?}(v)$$

$$\text{err} ::= \text{underflow}$$

$$c ::= p \mid \text{err}$$

$$o ::= v \mid \text{err}$$

$\rightsquigarrow \subseteq \text{REDEX} \times \text{PGM}$	Same formal schema as BA (over updated p)
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$\longrightarrow \subseteq \text{PGM} \times \text{CONFIG}$	Single-step Reduction
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$$\frac{r \rightsquigarrow p}{E[r] \longrightarrow E[p]} \qquad \frac{}{E[\text{pred}(0)] \longrightarrow \text{underflow}}$$

$\models \dots \subseteq \text{TERM} \times \text{TYPE}$ **Semantic Typing** (Q: Should t be statically typed? Interesting implications!)

$\models t : \text{Bool}$ if and only if $t \longrightarrow^* b$ or $t \longrightarrow^* \text{underflow}$

$\models t : \text{Nat}$ if and only if $t \longrightarrow^* n$ or $t \longrightarrow^* \text{underflow}$

$\vdash \dots \subseteq \text{TERM} \times \text{TYPE}$ **Syntactic Typing**

$$\frac{}{\vdash \text{true} : \text{Bool}} \quad \frac{}{\vdash \text{false} : \text{Bool}} \quad \frac{\vdash t_1 : \text{Bool} \quad \vdash t_2 : T \quad \vdash t_3 : T}{\vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T} \quad \frac{}{\vdash n : \text{Nat}} \quad \frac{\vdash t : \text{Nat}}{\vdash \text{succ}(t) : \text{Nat}}$$

$$\frac{\vdash t : \text{Nat}}{\vdash \text{pred}(t) : \text{Nat}} \quad \frac{\vdash t : \text{Nat}}{\vdash \text{zero?}(t) : \text{Bool}}$$

Evaluator $\boxed{\text{eval}_{TBA} : \text{PGM} \rightarrow \text{OBS}}$

$$\text{eval}_{TBA}(p) = b \text{ if } p \longrightarrow^* b$$

$$\text{eval}_{TBA}(p) = n \text{ if } p \longrightarrow^* n$$

$$\text{eval}_{TBA}(p) = \text{underflow} \text{ if } p \longrightarrow^* \text{underflow}$$

Safety

Conjecture 3 (Progress). For all $p \in \text{PGM}$ one of the following is true:

1. $p \in \text{VALUE}$;
2. $p \longrightarrow p'$ for some $p' \in \text{PGM}$;
3. $p \longrightarrow \text{underflow}$.

Conjecture 4 (Preservation). If $\vdash p_1 : T$ and $p_1 \longrightarrow p_2$ then $\vdash p_2 : T$.

Conjecture 5 (Semantic Type Soundness). If $\vdash t : T$ then $\models t : T$.

2 MBA: Mixed Boolean and Arithmetic Language

Program Static Syntax

$s \in \text{SBA}$, $d \in \text{DBA}$, $n, ns, nd \in \mathbb{N}^*$, $b, bs, bd \in \mathbb{B}^*$, \star : A bit sloppy perhaps
 $ps \in \text{SPGM} = \{s \in \text{SBA} \mid \exists T \in \text{TYPE}. \vdash s : T\}$, $pd \in \text{DPGM} = \{d \in \text{DBA} \mid \vdash d \checkmark\}$, $p \in \text{PGM} = ps$
 $s ::= \text{true} \mid \text{false} \mid \text{if } s \text{ then } s \text{ else } s$
 $\quad \mid ns \mid \text{succ}(s) \mid \text{pred}(s) \mid \text{zero?}(s) \mid [d]$
 $d ::= \text{true} \mid \text{false} \mid \text{if } d \text{ then } d \text{ else } d$
 $\quad \mid nd \mid \text{succ}(d) \mid \text{pred}(d) \mid \text{zero?}(d) \mid [s]$
 $bs ::= \text{true} \mid \text{false}$
 $bd ::= \text{true} \mid \text{false}$
 $b ::= bs \mid bd$

$\vdash \dots \subseteq \text{SBA} \times \text{TYPE}$ $\vdash \dots \checkmark \subseteq \text{DBA}$	Syntactic Typing
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$\frac{}{\vdash \text{true} : \text{Bool}}$	$\frac{}{\vdash \text{false} : \text{Bool}}$	$\frac{\vdash s_1 : \text{Bool} \quad \vdash s_2 : T \quad \vdash s_3 : T}{\vdash \text{if } s_1 \text{ then } s_2 \text{ else } s_3 : T}$	$\frac{}{\vdash ns : \text{Nat}}$	$\frac{\vdash s : \text{Nat}}{\vdash \text{succ}(s) : \text{Nat}}$	
	$\frac{\vdash s : \text{Nat}}{\vdash \text{pred}(s) : \text{Nat}}$	$\frac{\vdash s : \text{Nat}}{\vdash \text{zero?}(s) : \text{Bool}}$	$\frac{\vdash d \checkmark}{\vdash [d] : T}$		
$\frac{}{\vdash \text{true} \checkmark}$	$\frac{}{\vdash \text{false} \checkmark}$	$\frac{\vdash d_1 \checkmark \quad \vdash d_2 \checkmark \quad \vdash d_3 \checkmark}{\vdash \text{if } d_1 \text{ then } d_2 \text{ else } d_3 \checkmark}$	$\frac{}{\vdash nd \checkmark}$	$\frac{\vdash d \checkmark}{\vdash \text{succ}(d) \checkmark}$	$\frac{\vdash d \checkmark}{\vdash \text{pred}(d) \checkmark}$
		$\frac{\vdash d \checkmark}{\vdash \text{zero?}(d) \checkmark}$		$\frac{\vdash s : T}{\vdash [s] \checkmark}$	

Program Runtime Syntax

$Es \in \text{SECTXT}$, $Ed \in \text{DECTXT}$, $vs \in \text{SVALUE}$, $vd \in \text{DVALUE}$,
 $rs \in \text{SREDEX}$, $rd \in \text{DREDEX}$, $fs \in \text{SFAULTY}$, $fd \in \text{DFAULTY}$, $cs \in \text{SCONFIG}$, $ed \in \text{DCONFIG}$,
 $v \in \text{VALUE}$, $err \in \text{ERROR}$, $o \in \text{OBS}$
 $Es ::= \square \mid Es[\text{if } \square \text{ then } ps \text{ else } ps] \mid Es[\text{succ}(\square) \mid Es[\text{pred}(\square)] \mid Es[\text{zero?}(\square)] \mid Ed[[\square]]$
 $Ed ::= \square \mid Ed[\text{if } \square \text{ then } ps \text{ else } ps] \mid Ed[\text{succ}(\square) \mid Ed[\text{pred}(\square)] \mid Ed[\text{zero?}(\square)] \mid Es[[\square]]$
 $vs ::= bs \mid ns \mid [bd] \mid [nd]$
 $vd ::= bd \mid nd \mid [bs] \mid [ns]$
 $rs ::= [RG : \text{Exercise!}]$
 $rd ::= [RG : \text{Exercise!}]$
 $fs ::= [RG : \text{Exercise!}]$
 $fd ::= [RG : \text{Exercise!}]$
 $err ::= \text{mismatch} \mid \text{underflow}$
 $v ::= b \mid n$
 $cs ::= ps \mid err$
 $o ::= v \mid err$

$\rightsquigarrow \subseteq \text{REDEX} \times \text{PGM}$	Notions of Reduction
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$[RG : \text{Exercise! Here's old ones:}]$

$\text{if true then } p_2 \text{ else } p_3 \rightsquigarrow p_2$
 $\text{if false then } p_2 \text{ else } p_3 \rightsquigarrow p_3$
 $\text{pred}(n) \rightsquigarrow n - 1 \quad \text{if } n > 0$
 $\text{zero?}(0) \rightsquigarrow \text{true}$
 $\text{zero?}(n) \rightsquigarrow \text{false} \quad \text{if } n > 0$
 $\text{succ}(n) \rightsquigarrow n + 1$

$\longrightarrow \subseteq \text{PGM} \times \text{CONFIG}$	Single-step Reduction
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$[RG : \text{Exercise! Here's old ones:}]$

$\frac{r \rightsquigarrow p}{E[r] \longrightarrow E[p]}$
$\frac{}{E[f] \longrightarrow \text{mismatch}}$
$\frac{}{E[\text{pred}(0)] \longrightarrow \text{underflow}}$

$\models \cdot : \cdot \subseteq \text{TERM} \times \text{TYPE}$ **Semantic Typing** [RG: Exercise! Here's an old one:]

$\models t : \text{Bool}$ if and only if $t \longrightarrow^* b$ or $t \longrightarrow^* \text{underflow}$

$\models t : \text{Nat}$ if and only if $t \longrightarrow^* n$ or $t \longrightarrow^* \text{underflow}$

$\boxed{\text{eval}_{MBA} : \text{PGM} \rightarrow \text{OBS}}$

$\text{eval}_{MBA}(p) = b$ if $p \longrightarrow^* \text{bs}$

$\text{eval}_{MBA}(p) = b$ if $p \longrightarrow^* [\text{bd}]$

$\text{eval}_{MBA}(p) = n$ if $p \longrightarrow^* \text{ns}$

$\text{eval}_{MBA}(p) = n$ if $p \longrightarrow^* [\text{nd}]$

$\text{eval}_{MBA}(p) = \text{mismatch}$ if $p \longrightarrow^* \text{mismatch}$

$\text{eval}_{MBA}(p) = \text{underflow}$ if $p \longrightarrow^* \text{underflow}$

Safety

Conjecture 6 (Progress). For all $ps \in \text{SPGM}$ one of the following is true:

1. $ps \in \text{SVALUE}$;
2. $ps \longrightarrow ps'$ for some $ps' \in \text{SPGM}$;
3. $ps \longrightarrow \text{err}$ for some $\text{err} \in \text{ERROR}$.

Conjecture 7 (Preservation). If $\vdash p_1 : T$ and $p_1 \longrightarrow p_2$ then $\vdash p_2 : T$.

Conjecture 8 (Semantic Type Soundness). If $\vdash t : T$ then $\models t : T$.

3 mMBA: Minimal Mixed Boolean and Arithmetic Language

Missing!