Linear and Dependent Types, Part 1

Neel Krishnaswami
University of Cambridge

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1 Papers

- *Linear logic, monads, and the lambda calculus*. Nick Benton, Philip Wadler. 11th IEEE Symposium on Logic in Computer Science (LICS). New Brunswick, New Jersey, July 1996.
- *L^3 : A Linear Language with Locations*. Amal Ahmed, Matthew Fluet, Greg Morrisett. Fundamenta Informaticae XXI (2001). 1001 – 1053.
- *Integrating Linear and Dependent Types*. Neelakantan R. Krishnaswami, Pierre Pradic, Nick Benton. ACM SIGPLAN Symposium on Principles of Programming Programming Languages (POPL). Mumbai, India. January 2015.

2 Propositional Adjoint Logic

2.1 Syntax

$$\begin{array}{llcl}
\text{Intuitionistic Types} & X & ::= & 1 \mid X \times Y \mid X \to Y \mid G\,A \\
\text{Linear Types} & A & ::= & I \mid A \otimes B \mid A \multimap B \mid F\,X \\
\\
\text{Intuitionistic Contexts} & \Gamma & ::= & \cdot \mid \Gamma, x : X \\
\text{Linear Contexts} & \Delta & ::= & \cdot \mid \Delta, a : A \\
\\
\text{Intuitionistic Terms} & e & ::= & () \mid (e, e') \mid \mathsf{fst}\,e \mid \mathsf{snd}\,e \mid \lambda x.\,e \mid e\;e' \mid \mathsf{G}(t) \mid x \\
\text{Linear Terms} & t & ::= & () \mid \mathsf{let}\;() = t\;\mathsf{in}\;t' \mid (t, t') \mid \mathsf{let}\;(x, y) = t\;\mathsf{in}\;t' \\
 & & \mid & \lambda a.\,t \mid t\;t' \mid \mathsf{F}(e) \mid \mathsf{let}\;\mathsf{F}(x) = t\;\mathsf{in}\;t' \mid \mathsf{run}(e) \mid a
\end{array}$$

2.2 Rules

$$\boxed{\Gamma \vdash e : X}$$

$$\frac{}{\Gamma \vdash () : 1}\ 1\mathrm{I} \qquad \text{(no 1E)}$$

$$\frac{\Gamma \vdash e : X \qquad \Gamma \vdash e' : Y}{\Gamma \vdash (e, e') : X \times Y}\ \times\mathrm{I} \qquad \frac{\Gamma \vdash e : X \times Y}{\Gamma \vdash \mathsf{fst}(e) : X}\ \times\mathrm{E}_1 \qquad \frac{\Gamma \vdash e : X \times Y}{\Gamma \vdash \mathsf{snd}(e) : X}\ \times\mathrm{E}_2$$

$$\frac{\Gamma, x : X \vdash e : Y}{\Gamma \vdash \lambda x.\, e : X \to Y}\ \to\mathrm{I} \qquad \frac{\Gamma \vdash e : X \to Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash e\ e' : Y}\ \to\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash t : A}{\Gamma \vdash \mathsf{G}(t) : \mathsf{G}(A)}\ \mathsf{GI} \qquad \frac{x : X \in \Gamma}{\Gamma \vdash x : X}\ \textsc{VarI}$$

$$\boxed{\Gamma; \Delta \vdash t : A}$$

$$\frac{}{\Gamma; \cdot \vdash () : \mathsf{I}}\ \mathsf{II} \qquad \frac{\Gamma; \Delta \vdash t : \mathsf{I} \qquad \Gamma; \Delta' \vdash t' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ () = t\ \mathsf{in}\ t' : C}\ \mathsf{IE}$$

$$\frac{\Gamma; \Delta \vdash t : A \qquad \Gamma; \Delta' \vdash t' : B}{\Gamma; \Delta, \Delta' \vdash (t, t') : A \otimes B}\ \otimes\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash t : A \otimes B \qquad \Gamma; \Delta', a : A, b : B \vdash t' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ (a, b) = t\ \mathsf{in}\ t' : C}\ \otimes\mathrm{E}$$

$$\frac{\Gamma; \Delta, a : A \vdash t : B}{\Gamma; \Delta \vdash \lambda a.\, t : A \multimap B}\ \multimap\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash t : A \multimap B \qquad \Gamma; \Delta' \vdash t' : A}{\Gamma; \Delta, \Delta' \vdash t\ t' : B}\ \multimap\mathrm{E}$$

$$\frac{\Gamma \vdash e : X}{\Gamma; \cdot \vdash \mathsf{F}(e) : \mathsf{F}(X)}\ \mathsf{FI} \qquad \frac{\Gamma; \Delta \vdash t : \mathsf{F}(X) \qquad \Gamma, x : X; \Delta' \vdash t' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ \mathsf{F}(x) = t\ \mathsf{in}\ t' : C}\ \mathsf{FE}$$

$$\frac{\Gamma \vdash e : \mathsf{G}(A)}{\Gamma; \cdot \vdash \mathsf{run}(e) : A}\ \mathsf{GE} \qquad \frac{}{\Gamma; a : A \vdash a : A}\ \textsc{VarL}$$

2.3 Substitution Properties

Lemma 1. *(Weakening) We have that:*

- *If* $\Gamma \vdash e : Y$ *then* $\Gamma, x : X \vdash e : Y$.
- *If* $\Gamma; \Delta \vdash t : A$ *then* $\Gamma, x : X; \Delta \vdash t : A$.

Theorem 1. *(Substitution) We have that:*

- *If* $\Gamma \vdash e : X$ *and* $\Gamma, x : X \vdash e' : Y$ *then* $\Gamma \vdash [e/x]e' : Y$.
- *If* $\Gamma \vdash e : X$ *and* $\Gamma, x : X; \Delta \vdash t : A$ *then* $\Gamma; \Delta \vdash [e/x]t : A$.
- *If* $\Gamma; \Delta \vdash t : A$ *and* $\Gamma; \Delta', a : A \vdash t' : B$ *then* $\Gamma \vdash [t/a]t' : B$.