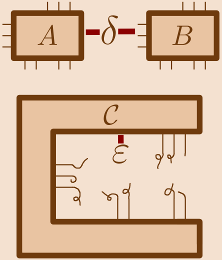
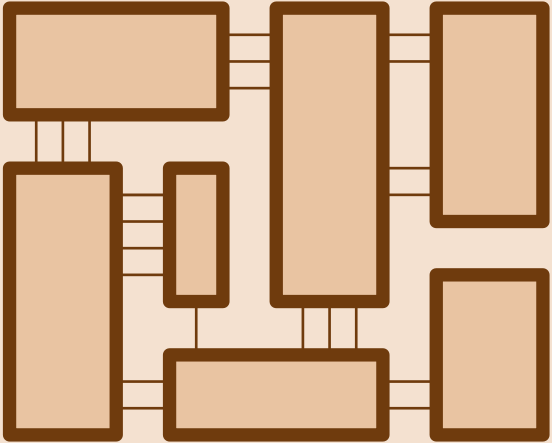
ON DIFFERENTIAL PROGRAM SEMANTICS



Ugo Dal Lago

Department of
Computer Science and Engineering
University of Bologna

PROGRAM SEMANTICS



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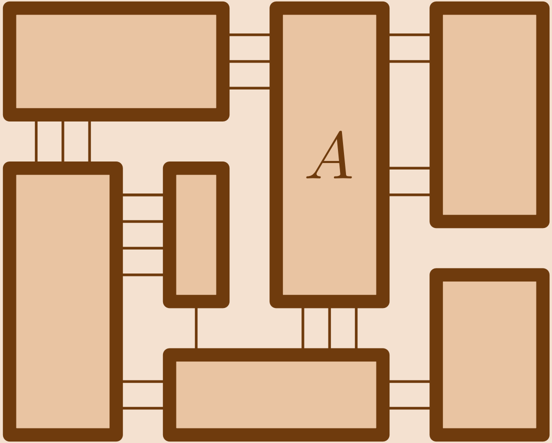

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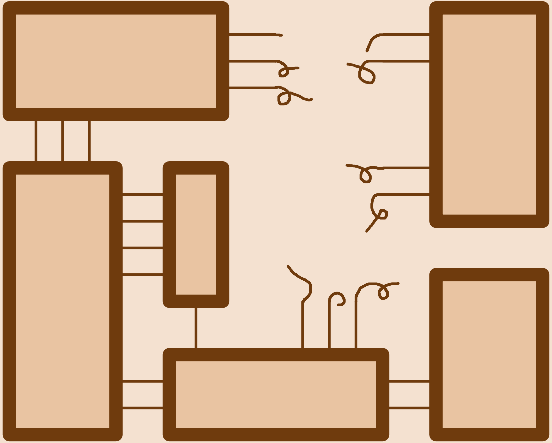

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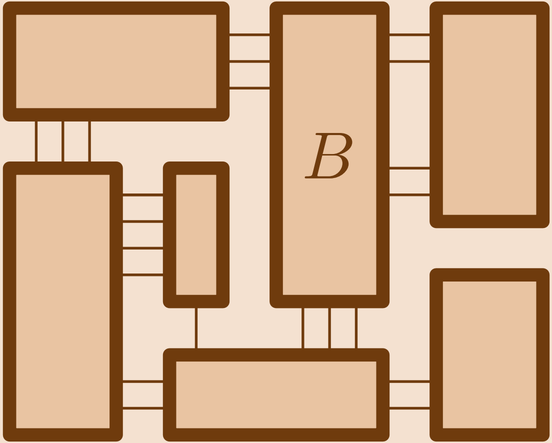

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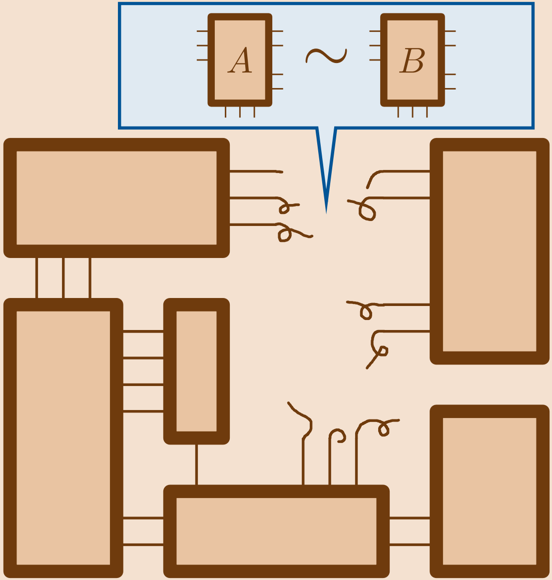
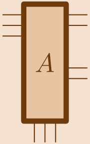
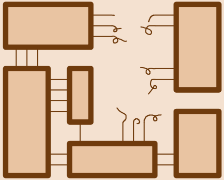


DENOTATIONAL SEMANTICS



$$A^* \in \tau$$



$$\mathcal{C}^* : \tau \to \rho$$

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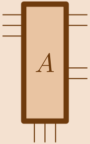

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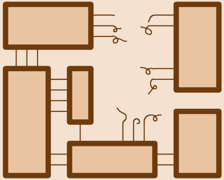

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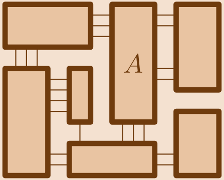
$$\mathcal{C}^*(A^*)$$



A Recipe

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1. Given a programming language, define a **type system**.

$$\tau ::= \texttt{NAT} \mid \tau \to \rho$$

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2. Define a **family of relations** between programs, indexed by types, and following their structure.

$$M \mathrel{\mathcal{R}_{\tau\to\rho}} N \quad \text{iff} \quad (M L) \mathrel{\mathcal{R}_{\rho}} (N P) \quad \text{whenever} \quad L \mathrel{\mathcal{R}_{\tau}} P.$$

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3. Prove the **Fundamental Theorem**

$$\vdash M : \tau \implies M \mathrel{\mathcal{R}_\tau} M$$

A Recipe

1. Giv…

- Works extremely well for typed programming languages.
- Along the years, adapted to language with non-inductive type structures, concurrent programming languages, etc.
- Often proved fully abstract with respect to contextual equivalence:

$$M \equiv N \Longleftrightarrow \forall C.Obs(C[M]) = Obs(C[N]).$$

- Due to Plotkin, but later studied by many others.

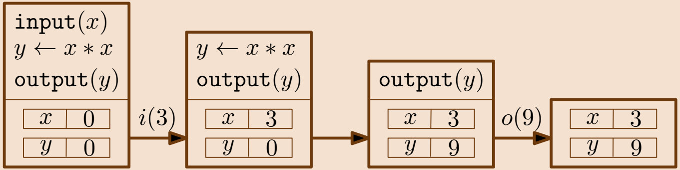
2. Defi… …nd follo…

3. Prov…

$$\vdash M : \tau \implies M \mathcal{R}_\tau M$$

A Recipe

1. See program (configurations) as states of a **labelled transition system**.



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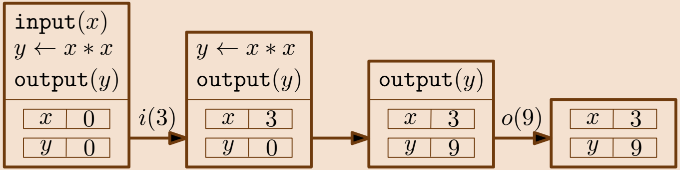

2. Define a notion of **bisimulation relation**.

$$(M \mathcal{R} N) \wedge (M \xrightarrow{\ell} L) \implies \exists P.(N \xrightarrow{\ell} P) \wedge (L \mathcal{R} P),$$

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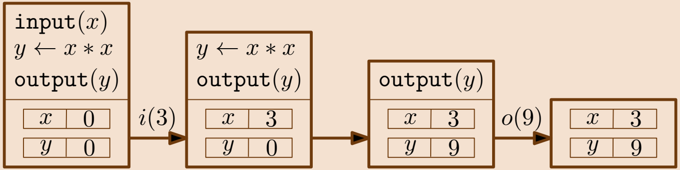

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3. Prove that **bisimilarity is a congruence**.

$$M \sim N \implies C[M] \sim C[N].$$

A Recipe

1. See program (configurations) as states of a **labelled transition system**.

`input`(x)
$y \leftarrow x * x$

$y \leftarrow x * x$

- Very successful in the absence of types, and in a concurrent scenario.
- Becomes extremely useful in practice when coupled with so-called up-to techniques.
- Incepted into program semantics by Milner.

2. Defi

$$(N \mathrel{\mathcal{R}} M) \wedge (N \xrightarrow{\ell} P) \implies \exists L.(M \xrightarrow{\ell} L) \wedge (L \mathrel{\mathcal{R}} P).$$

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1. Types are interpreted as **games**, namely sets of sequences of **moves**.

$$GAME(\texttt{NAT}) = \{\varepsilon, q, q \cdot 0, q \cdot 1, q \cdot 2, \cdots\}$$

$$GAME(\texttt{NAT} \to \texttt{BOOL}) = \{\varepsilon, q_{\texttt{BOOL}}, q_{\texttt{BOOL}} \cdot \mathit{True}, q_{\texttt{BOOL}} \cdot \mathit{False}, q_{\texttt{BOOL}} \cdot q_{\texttt{NAT}}, \cdots\}$$

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2. Programs are interpreted as **strategies**, namely functions which returns *the next move* from the history.

$$f = STRATEGY(\lambda x.\texttt{if}\ (x > 0)\ \texttt{then}\ \mathit{True}\ \texttt{else}\ \mathit{False})$$

$$f(q_{\texttt{BOOL}}) = q_{\texttt{NAT}}; \qquad f(q_{\texttt{BOOL}} \cdot q_{\texttt{NAT}} \cdot 3) = \mathit{True}$$

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3. Computation is seen as the **interaction** between a program, modeled as a strategy, and its environment, itself seen as a (possibly nondeterministic) strategy, called the **observer**.

A Recipe

1. Types are interpreted as **games**, na[…]ely sets of sequences of **moves**.

   GA[…] […], …}

2. Pro[…] […]rns *the* […]

- The equivalence induced by interpreting programs as strategies has been proved to *coincide* with contextual equivalence in many cases.
- It has been proved to be adaptable to many different kinds of programming languages, and to concurrent languages in particular.
- There is a symmetry between the program and the environment, which can be seen itself as a strategy.

3. Co[…] […] **interaction** […] as a strategy, and its environment, itself seen as a (possibly nondeterministic) strategy, called the **observer**.

A Story of Successes…

A Story of Successes. . .

. . . is This The End?

A Story of Successes. . .

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Definitely No!

- Can we turn logical relations into a *metric*?

$$\mathcal{R}_\tau \subseteq \Lambda_\tau \times \Lambda_\tau \qquad \longmapsto \qquad \Delta_\tau : \Lambda_\tau \times \Lambda_\tau \to \mathbb{R}^\infty_{\geq 0}$$

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- We can answer this question affirmatively [RP2010]! Define a **family of metrics**, indexed by types, as follows:

$$\Delta_{\tau\to\rho}(M,N) \leq k \quad \text{iff} \quad \Delta_\rho(ML,NP) \leq k + \Delta_\tau(L,P) \text{ whenever } L,P:\tau$$

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- The **Fundamental Theorem** can be extended naturally:

$$\Delta_\tau(M,M) = 0 \text{ whenever } M : \tau$$

- Can we

  $: \Lambda_\tau \times \Lambda_\tau \to \mathbb{R}^{\infty}_{\geq 0}$

- This only works in a **affine** setting, so not in the case of the STλC.
- Functions on the reals we start from must be Lipschitz
- Categorically, this corresponds to the fact that metric spaces with Lipchitz maps form a SMCC, but not a CCC.

- We can 10]! Define a **family of metric**

  $\Delta_{\tau\to\rho}$ (L, P) whenever $L, P : \tau$

- The **Fundamental Theorem** can be extended naturally:

$$\Delta_\tau(M, M) = 0 \text{ whenever } M : \tau$$

EXAMPLE

```
def A(x):
  return sin(x)
```

```
def B(x):
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```

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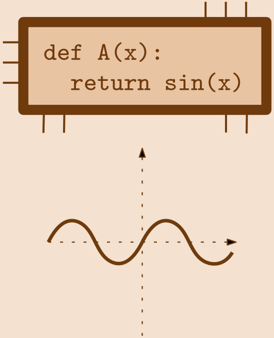

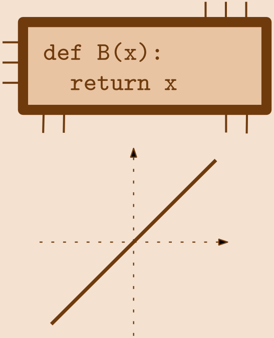

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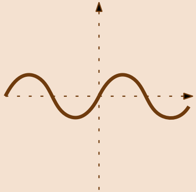

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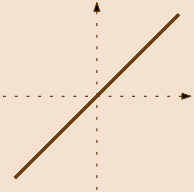

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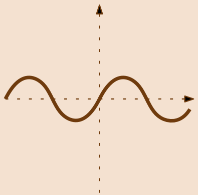

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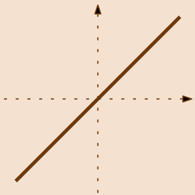

Not Equivalent

At **Infinite** Distance

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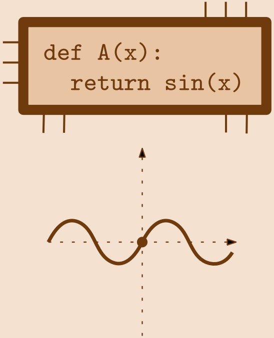

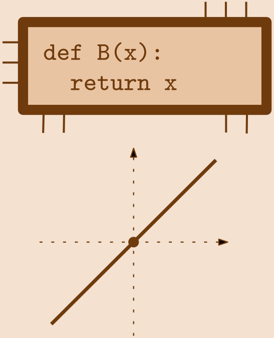

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But **Very Similar** if x is Close to 0

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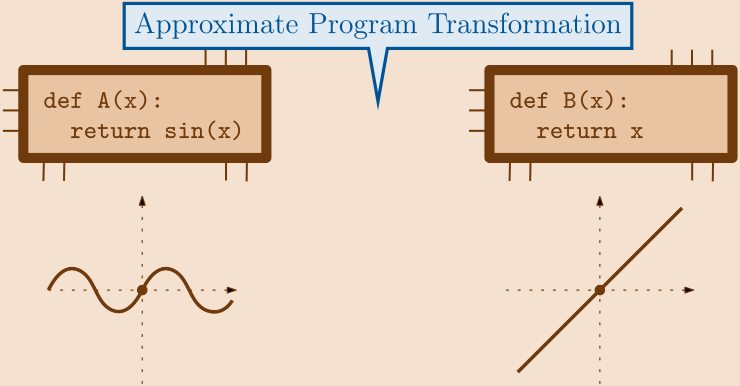

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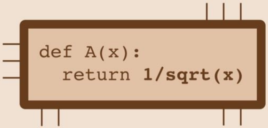
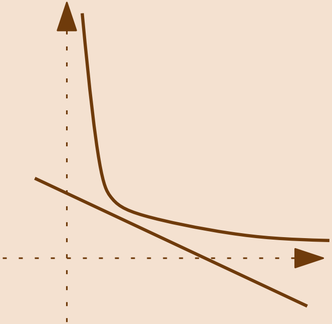
FURTHER EXAMPLES



FURTHER EXAMPLES

```
def A(x):
  return 1/sqrt(x)
```

```
def B(x):
  return 2.08-1.119*x
```



FURTHER EXAMPLES

Loop perforation

```
def sum_to_n(n):
  sum = 0
  for (i=0, i<n, i++)
    sum += i
  return sum
```

```
def sum_to_n(n):
  sum = 0
  for (i=0, i<n, i+=k)
    sum += i
  return sum
```

```
float Q_rsqrt( float number )
{
long i;
float x2, y;
const float threehalfs = 1.5F;
 x2 = number * 0.5F;
 y  = number;
 i  = * ( long * ) &y;
 i  = 0x5f3759df - ( i >> 1 );
 y  = * ( float * ) &i;
 y  = y * ( threehalfs - ( x2 * y * y ) );
return y;
```

Fast inverse square root

The **Difference** is the Meaning.

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The Context **Matters**.

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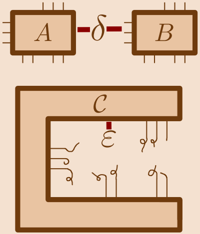

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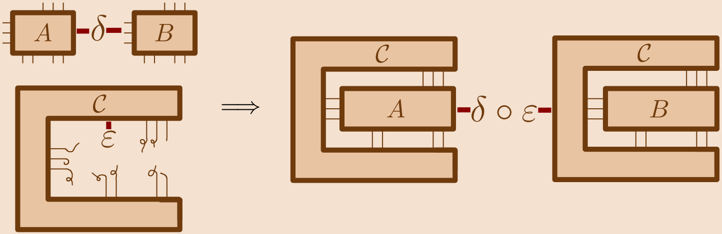

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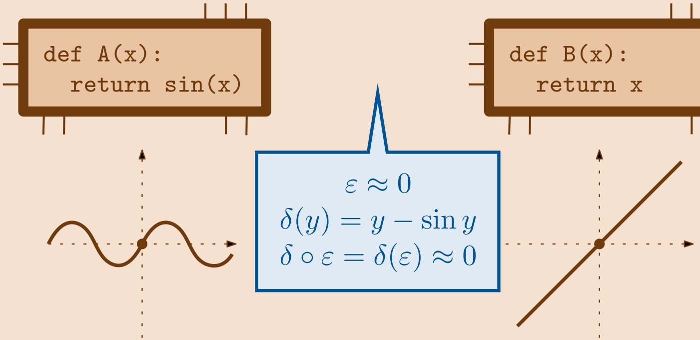

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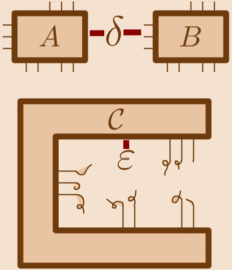

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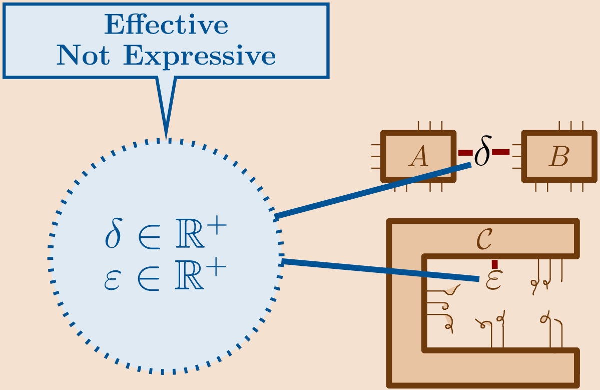

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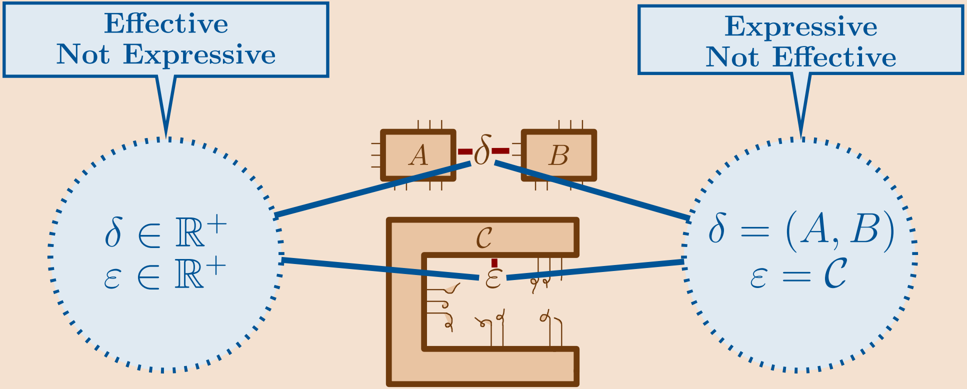

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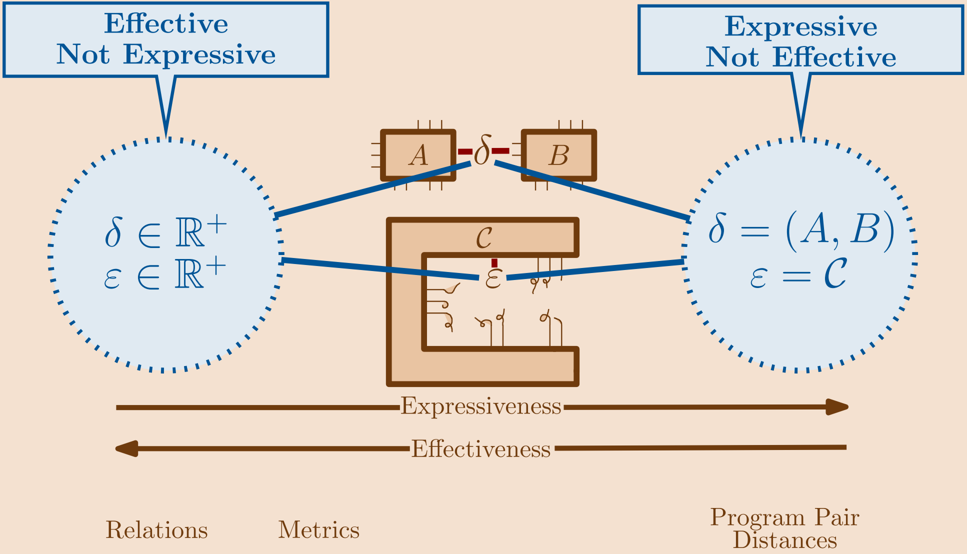

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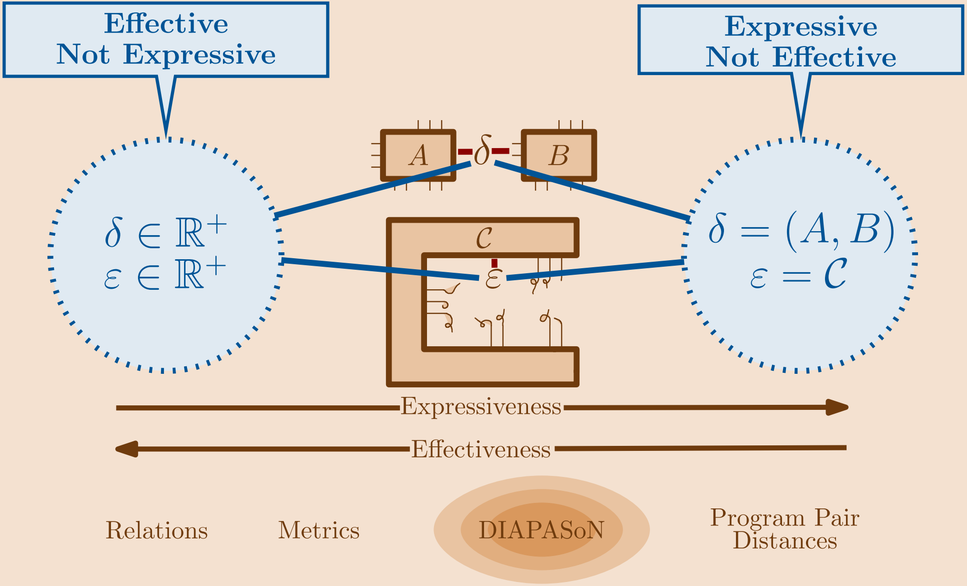

Differential Logical Relations

Types

$$\tau, \rho ::= REAL \;\Big|\; \tau \to \rho \;\Big|\; \tau \times \rho$$

Types

$$\tau, \rho ::= REAL \mid \tau \to \rho \mid \tau \times \rho$$

Typing Rules

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{}{\Gamma \vdash r : REAL} \qquad \frac{f_n \in \mathcal{F}_n}{\Gamma \vdash f_n : REAL^n \to REAL} \qquad \frac{\Gamma, x : \tau \vdash M : \rho}{\Gamma \vdash \lambda x.M : \tau \to \rho}$$

$$\frac{\Gamma \vdash M : \tau \to \rho \quad \Gamma \vdash N : \tau}{\Gamma \vdash MN : \rho} \qquad \frac{\Gamma \vdash M : \tau \quad \Gamma \vdash N : \rho}{\Gamma \vdash \langle M, N \rangle : \tau \times \rho} \qquad \frac{}{\Gamma \vdash \pi_1 : \tau \times \rho \to \tau} \qquad \frac{}{\Gamma \vdash \pi_2 : \tau \times \rho \to \rho}$$

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A TOY LANGUAGE

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Denotational Semantics

$$[\![REAL]\!] = \mathbb{R}; \qquad [\![\tau \to \rho]\!] = [\![\tau]\!] \to [\![\rho]\!]; \qquad [\![\tau \times \rho]\!] = [\![\tau]\!] \times [\![\rho]\!].$$

THE DEFINITION

Distance Spaces

$(\!|REAL|\!) = \mathbb{R}^{\infty}_{\geq 0};$ $(\!|\tau \to \rho|\!) = [\![\tau]\!] \times (\!|\tau|\!) \to (\!|\rho|\!);$ $(\!|\tau \times \rho|\!) = (\!|\tau|\!) \times (\!|\rho|\!);$

Distance Spaces

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DLRs as Ternary Relations

$$\begin{aligned}
\delta_{REAL}(M, r, N) &\Leftrightarrow |NF(M) - NF(N)| \leq r; \\
\delta_{\tau\times\rho}(M, (d_1, d_2), N) &\Leftrightarrow \delta_\tau(\pi_1 M, d_1, \pi_1 N) \wedge \delta_\rho(\pi_2 M, d_2, \pi_2 N) \\
\delta_{\tau\to\rho}(M, d, N) &\Leftrightarrow (\forall V \in CV(\tau).\ \forall x \in (\!|\tau|\!).\ \forall W \in CV(\tau). \\
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DLRs as Ternary Relations

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\delta_{REAL}(M, r, N) &\Leftrightarrow |NF(M) - NF(N)| \leq r; \\
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\end{aligned}$$

Theorem (Fundamental Lemma, Version I)

For every $\vdash M : \tau$, *there is* $d \in (\!|\tau|\!)$ *such that* $\delta_\tau(M, d, M)$.

BACK TO THE EXAMPLE

Claim

$$\delta_{REAL \to REAL}(M_{ID}, \lambda\!\!\lambda \langle x, y \rangle . y + |x - \sin x|, M_{SIN})$$

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Proof.

Consider any pairs of real numbers $r, s \in \mathbb{R}$ such that $|r - s| \leq \varepsilon$, where $\varepsilon \in \mathbb{R}^{\infty}_{\geq 0}$. We have that:

$$\begin{aligned}
|\sin r - s| &= |\sin r - r + r - s| \leq |\sin r - r| + |r - s| \\
&\leq |\sin r - r| + \varepsilon = f(r, \varepsilon) \\
|\sin s - r| &= |\sin s - \sin r + \sin r - r| \\
&\leq |\sin s - \sin r| + |\sin r - r| \leq |s - r| + |\sin r - r| \\
&\leq \varepsilon + |\sin r - r| = f(r, \varepsilon).
\end{aligned}$$

□

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$$M_{ID} = \lambda x.x$$

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- (The function induced by) δ_ρ, indeed, does not satisfy the reflexivity axiom, and as a consequence does not have the structure of a metric.
- It can however be given the structure of a generalized metric domain.

Hereditarily Null Distances

$$(\!|REAL|\!)^0 = \{0\} \qquad (\!|\tau \times \rho|\!)^0 = (\!|\tau|\!)^0 \times (\!|\rho|\!)^0$$

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$$(\!|REAL|\!)^{<\infty} = \mathbb{R}_{\geq 0}; \qquad (\!|\tau \times \rho|\!)^{<\infty} = (\!|\tau|\!)^{<\infty} \times (\!|\rho|\!)^{<\infty};$$

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Lemma

Whenever $\vdash M, N : \tau$, M *is logically related to* N *iff* $\delta_\tau(M, d, N)$ *where* $d \in (\!|\tau|\!)^0$.

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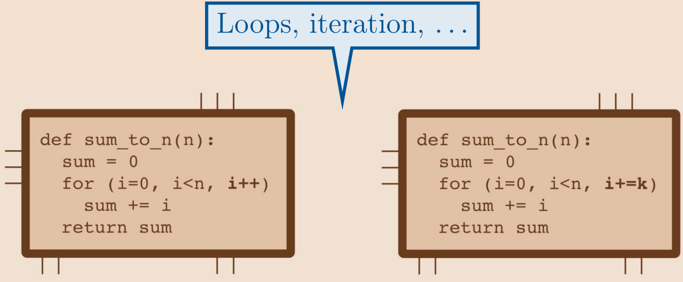
Lemma

Whenever $\vdash M, N : \tau$, M is logically related to N iff $\delta_\tau(M, d, N)$ where $d \in (\!|\tau|\!)^0$.

Theorem (Fundamental Lemma, Version II)

For every $\vdash M : \tau$, there is $d \in (\!|\tau|\!)^{<\infty}$ such that $\delta_\tau(M, d, M)$.

RECURSION


Loops, iteration, . . .
def sum_to_n(n):
sum = 0
for (i=0, i<n, i++)
sum += i
return sum
def sum_to_n(n):
sum = 0
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Adding Recursion

$$\frac{\Gamma, f : \tau \to \rho, x : \tau \vdash M : \rho}{\Gamma \vdash \texttt{fix}(f, x).M : \tau \to \rho}$$

Step-indexed DLRs

$$\delta_{REAL}(n, V, r, W) \Leftrightarrow |V - W| \leq r;$$

$$\vdots$$

$$\begin{aligned}\delta_{\tau\to\rho}(n, V, d, W) \Leftrightarrow {} & (\forall k < n, \forall U \in CV(\tau).\ \forall x \in (\!|\tau|\!).\ \forall Z \in CV(\tau).\\ & \delta_\tau(k, U, x, Z) \Rightarrow \delta_\rho(k, VU, d([\![U]\!], x), WZ)\\ & \wedge \delta_\rho(k, VZ, d([\![U]\!], x), WU));\end{aligned}$$

$$\begin{aligned}\delta_\tau(n, M, d, N) \Leftrightarrow {} & \forall k < n.\ (M \Downarrow_k V \wedge N \Downarrow W \Rightarrow \delta_\tau(n-k, V, d, W) \wedge\\ & N \Downarrow_k W \wedge M \Downarrow V \Rightarrow \delta_\tau(n-k, V, d, W)).\end{aligned}$$

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Theorem (Fundamental Lemma, Version III)

For every $\vdash M : \tau$ *and* $n \geq 0$*, there is* $d \in (\!|\tau|\!)$ *such that* $\delta_\tau(n, M, d, M)$.

EFFECTS

```
float Q_rsqrt( float number )
{
long i;
float x2, y;
const float threehalfs = 1.5F;
 x2 = number * 0.5F;
 y  = number;
 i  = * ( long * ) &y;
 i  = 0x5f3759df - ( i >> 1 );
 y  = * ( float * ) &i;
 y  = y * ( threehalfs - ( x2 * y * y ) );
return y;
```

- Imperative features
- Errors
- Randomness
- IO

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Adding Effects

$$\tau \ ::= \ \cdots \mid T(\tau)$$

- Global states
- Distribution
- Powerset

EFFECTS

Effectful Distance Spaces

$$(\!|REAL|\!) = \mathbb{R}^{\infty}_{\geq 0}; \quad (\!|\tau \to \rho|\!) = [\![\tau]\!] \times (\!|\tau|\!) \to (\!|\rho|\!); \quad (\!|\tau \times \rho|\!) = (\!|\tau|\!) \times (\!|\rho|\!); \quad (\!|T(\tau)|\!) = U(\!|\tau|\!)$$

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- **Randomness.** $T = U = \mathcal{D}$ (**distribution monad**)
- **Output.**
  - $T = \Sigma^* \times -$ (**output monad**);
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DLRs as Ternary Dependent Relations

$$\delta_A \in \prod_{a \in A} 2^{\Delta(a) \times A}$$

$\subseteq (\!|A|\!)$: Allowed distances for $a \in A$

Denotational Semantics

- Can we replace DLRs with **metrics**?

$$d \; : \; [\![\sigma]\!] \times [\![\sigma]\!] \longrightarrow (\!|\sigma|\!)$$

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- In usual metric semantics each type σ is associated with a metric $d : [\![\sigma]\!] \times [\![\sigma]\!] \longrightarrow \mathbb{R}^{\infty}_{\geq 0}$ with a **fixed** distance space, $\mathbb{R}^{\infty}_{\geq 0}$.

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- However, these semantics cannot account for STλC (they are not **cartesian closed**)
- By letting the distance spaces $(\!|\sigma|\!)$ *depend on* σ the picture changes

- Suppose that

$$(\!|REAL \to REAL|\!) = (\mathbb{R}^{\infty}_{\geq 0})^{\mathcal{I}(\mathbb{R})}$$

where $\mathcal{I}(\mathbb{R}) = \{\text{compact intervals of } \mathbb{R}\}$.

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- Then $p \;:\; [\![REAL \to REAL]\!] \times [\![REAL \to REAL]\!] \longrightarrow (\mathbb{R}^{\infty}_{\geq 0})^{\mathcal{I}(\mathbb{R})}$, where $p(f, g)(I) \;=\; \mathsf{diam}(f(I) \cup g(I))$.

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- This way we get a **partial metric space**:

$$p(f,f) \leq p(f,g), p(g,f)$$
$$p(f,g) = p(g,f)$$
$$p(f,g) = p(f,f) = p(g,g) \Rightarrow f = g$$
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- And out of it, a proper **metric space**:

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- This way $\mathbb{R}$ is extended to all simple types.

CoindUction and Game Semantics

- Given an LTS $(A, \mathcal{LBL}, \rightarrow)$, we say that $\delta : A \times A \rightarrow \mathbb{R}^{\infty}_{\geq 0}$ is a **behavioural metric** iff

$$\delta(M,N) \geq Obs(M,N)$$

$$\delta(M,N) \geq \delta(L,P) \text{ whenever } M \xrightarrow{\ell} L \wedge N \xrightarrow{\ell} P$$

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- Variations along this theme for LTSs in various monadic flavours are very well known and studied.
- When applied to LTSs coming from higher-order languages, these suffer from the same problems as MLRs:
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  - The distance does not depend on the context.

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- The label ℓ is arbitrary.
- $\delta(M, N)$ can be much bigger than $\delta(L, P)$ for some ℓ.

- Rather than taking $\mathbb{R}^{\infty}_{\geq 0}$ as the codomain of δ, one could take a set $\mathbb{Q}$ such that, e.g.

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- Then, the bisimulation game could be made **contextual**:

$$\pi_1(\delta(M, N)) \geq Obs(M, N)$$

$$\pi_2(\delta(M, N), \ell) \geq \delta(L, P) \text{ whenever } M \xrightarrow{\ell} L \wedge N \xrightarrow{\ell} P$$

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- We can capture contextual bisimiliarty [Larsen85].
- The properties of the induced notion of distance are still being scrutinized.

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- The distance $\delta_\tau(f, g)$ between two such strategies can then be taken as $f \cup g$.

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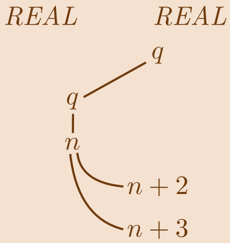
$$\delta_{REAL \to REAL}(\lambda x.x + 2, \lambda x.x + 3) \qquad =$$



- Quite suprisingly $\delta(f \circ h, g \circ k) = \delta(f, g) \circ \delta(h, k)$.
  - Very close to program pair distances.
  - How about playing on *abstract moves*?

Questions?