# FROM PROGRAM EQUIVALENCES TO PROGRAM METRICS

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OPLSS

RELATION

$R : X \mathrel{+\!\!\to} Y \qquad R \subseteq X \times Y$

$S : Y \mathrel{+\!\!\to} Z$

$$x\,(R;S)\,z \iff \exists y \in Y.\ \ x R y \wedge y S z$$

$\mathrm{REL}(X,Y) = \{ R \mid R : X \mathrel{+\!\!\to} Y \}$

COMPLETE LATTICE

$R : X \mathrel{+\!\!\to} Y$

$R^T = \{ (y,x) \mid (x,y) \in R \}$

REFLEXIVITY FOR $R \in \mathrm{REL}(X,X)$

$I \subseteq R$

TRANSITIVITY FOR $R \in \mathrm{REL}(X,X)$

$R;R \subseteq R$

SYMMETRY FOR $R \in \mathrm{REL}(X,X)$

$R^T \subseteq R$

# INTERMEZZO: $\Lambda^{ST}$

$$\sigma, \tau ::= \mathrm{BOOL} \mid \mathrm{UNIT} \mid \sigma \times \tau \mid \sigma \to \tau \qquad \text{TYPES}$$

$$v, w ::= x \mid \langle\rangle \mid \mathsf{true} \mid \mathsf{false} \mid \langle v, w\rangle \mid \lambda x.e \qquad \text{VALUES}$$

$$\begin{aligned} e, f ::= {} & \mathsf{return}\ v \mid \mathsf{if}\ v\ \mathsf{then}\ e\ \mathsf{else}\ f \mid \\ & \mathsf{proj}_l v \mid \mathsf{proj}_r v \mid v w \mid \\ & \mathsf{let}\ x = e\ \mathsf{in}\ f \qquad \text{COMPUTATIONS} \end{aligned}$$

$$\mathsf{if}\ e\ \mathsf{then}\ f\ \mathsf{else}\ g \equiv \mathsf{let}\ x = e\ \mathsf{in}\ \mathsf{if}\ x\ \mathsf{then}\ f\ \mathsf{else}\ g$$

$$e[v/x] \qquad w[v/x]$$

CAPTURE-AVOIDING SUBSTITUTIONS

ENVIRONMENT $\Gamma$ = PARTIAL FUNCTION FROM VARIABLES TO TYPES

$$\Gamma \vdash^{v} v : \sigma \qquad \Gamma \vdash^{\Lambda} e : \sigma$$

JUDGMENTS FOR VALUES

AND COMPUTATIONS

$$\frac{\Gamma \vdash^{\Lambda} e : \sigma \qquad \Gamma, x:\sigma \vdash^{\Lambda} f : \tau}{\Gamma \vdash^{\Lambda} \mathrm{let}\ x = e\ \mathrm{in}\ f : \tau}$$

. . . .

$$\frac{\Gamma \vdash^{\mathcal{V}} v : \sigma \to \tau \qquad \Gamma \vdash^{\mathcal{V}} w : \sigma}{\Gamma \vdash^{\Lambda} v\,w : \tau}$$

$\Lambda_\sigma \leftarrow$ OPEN COMPUT. $\qquad$ OPEN VALUES $\to \mathcal{V}_\sigma$

$\Lambda_{\Gamma \vdash \sigma} \qquad\qquad \mathcal{V}_{\Gamma \vdash \sigma}$

$\Lambda^\bullet_\sigma \leftarrow$ CLOSED COMPUT. $\qquad$ CLOSED VALUES $\to \mathcal{V}^\bullet_\sigma$

$$[\![ - ]\!]^n_{\varepsilon} : \prod_\sigma \left( \Lambda^\bullet_\sigma \longrightarrow \mathcal{V}^\bullet_\sigma \right)$$

$$[\![ e ]\!]^0_\varepsilon = \bot$$

$$[\![ \mathrm{return}\ v ]\!]^{n+1}_\varepsilon = v$$

$$[\![ \mathrm{proj}_e \langle v, w \rangle ]\!]^{n+1}_\varepsilon = v$$

$$\vdots$$

$$[\![ \mathrm{let}\ x = e\ \mathrm{in}\ f ]\!]^{n+1}_\varepsilon = \begin{cases} [\![ f[v/x] ]\!]^n_\varepsilon & \text{if } [\![ e ]\!]^n_\varepsilon = v \\ \bot & \text{OTHERWISE} \end{cases}$$

LEMMA

THE SEQUENCE OF MAPS $\left(\llbracket - \rrbracket^{n}_{\mathcal{E}}\right)_{n \geq 0}$ IS AN $\omega$-CHAIN IN THE $\omega$-CPO $\Lambda^{\circ}_{\sigma} \rightharpoonup \mathcal{V}^{\circ}_{\sigma}$

THERE IS THUS AN UPPER BOUND, CALL IT $\llbracket - \rrbracket_{\mathcal{E}}$, NAMELY THE OPERATIONAL SEMANTICS OF $\Lambda^{ST}$