FROM PROGRAM EQUIVALENCES TO PROGRAM METRICS

JUNE 24 2021

OPLSS

$\Lambda, \mathcal{V}$ $\quad \Gamma \vdash^{\Lambda} e : \sigma$ $\quad \Gamma \vdash^{\mathcal{V}} v : \sigma$ $\quad [\![-]\!]_{\varepsilon}$

$[\![\mathrm{return}(\lambda x.x)]\!] = \overset{I}{\overset{\shortparallel}{\lambda x.x}}$

$[\![\mathrm{return}(II)]\!] = I$

$[\![\mathrm{return}(\lambda x.Ix)]\!] = \lambda x.Ix$

DESIDERATA

- EQUIVALENCES: COMPATIBILITY, ADEQUACY
- REFINEMENTS: COMPATIBILITY, PREADEQUACY

- <u>CLOSED TERM RELATION</u> $\mathcal{R} = (\mathcal{R}^{\Lambda}, \mathcal{R}^{\mathcal{V}})$ WHERE
  - $\mathcal{R}^{\Lambda}$ IS A FAMILY $\mathcal{R}^{\Lambda} = \{\mathcal{R}^{\Lambda}_{\sigma}\}$ OF RELATIONS WHERE $\mathcal{R}^{\Lambda}_{\sigma} : \Lambda^{\circ}_{\sigma} \nrightarrow \Lambda^{\circ}_{\sigma}$
  - $\mathcal{R}^{\mathcal{V}}$ ...

- <u>(OPEN) TERM RELATION</u>: TO EACH SEQUENT $\Gamma \vdash \sigma$ THE RELATION $\mathcal{R}$ ASSIGNS A RELATION $\mathcal{R}^{\Lambda}_{\Gamma \vdash \sigma}$ ON $\Lambda_{\Gamma \vdash \sigma}$ AND A RELATION $\mathcal{R}^{\mathcal{V}}_{\Gamma \vdash \sigma}$ ON $\mathcal{V}_{\Gamma \vdash \sigma}$. THESE RELATIONS MUST BE STABLE BY WEAKENING, E.G

OPEN TERM RELATION

$$\frac{\Gamma \vdash^{\Lambda} e \,\mathcal{R}\, f : \sigma}{\Gamma, \Delta \vdash e \,\mathcal{R}\, f : \sigma}$$

- ON (CLOSED) TERM RELATIONS, ONE CAN DEFINE THE FOLLOWING OPERATORS:
  - COMPOSITION, $;$
  - TRANSPOSITION, $(\cdot)^T$
  - UNION AND INTERSECTION
- CLOSED RESTRICTION OF A TERM RELATION R IS THE CLOSED TERM RELATION R^c

$$\frac{\cdot \vdash^{\Lambda} e\, R\, f : \sigma}{e\, R^c_\sigma\, f} \quad \ldots$$

- OPEN EXTENSION OF A CLOSED TERM RELATION R IS A TERM RELATION R° SUCH THAT

$$\frac{\forall \bar{v} : \bar{\sigma} \quad e[\bar{v}/\bar{x}]\, R\, f[\bar{v}/\bar{x}]}{\bar{x}, \bar{\sigma} \vdash^{\Lambda} e\, R^\circ\, f : \tau} \quad \ldots$$

- SUBSTITUTIVE EXTENSION OF A CTR R IS A TERM RELATION R^s SUCH THAT

$$\frac{\forall \bar{v}, \bar{w} : \bar{\sigma}.\ \bar{v}\, R^v_\sigma\, \bar{w} \implies e[\bar{v}/\bar{x}]\, R\, f[\bar{w}/\bar{x}]}{\bar{x}, \bar{\sigma} \vdash e\, R^s\, f : \sigma} \quad \ldots$$

· GIVEN TWO TERM RELATIONS R, S, ONE CAN DEFINE THE SUBSTITUTION $R[S]$

$$\frac{\bar{x}:\bar{\sigma}, \Gamma \vdash e \, R \, f : \sigma \qquad \Gamma \vdash^{\vee} \bar{v} \, S \, \bar{w} : \bar{\sigma}}{\Gamma \vdash^{\wedge} e[\bar{v}/\bar{x}] \, R[S] \, f[\bar{w}/\bar{x}] : \sigma}$$

⋮

IDENTITY RELATION

· <u>VALUE-SUBSTITUTIVE</u> $R[I] \subseteq R$

· <u>SUBSTITUTIVE</u> $R[R] \subseteq R$

· <u>COMPATIBILITY</u> $I[R] \subseteq R$

ALTERNATIVE DEFINITION: BY WAY OF THE COMPATIBILITY REFINEMENT $\hat{R}$ OF ANY RELATION R

$$\frac{\bar{e} \, R \, \bar{f}}{c(\bar{e}) \; \hat{R} \; c(\bar{f})} \qquad \hat{R} \subseteq R$$

ADEQUACY

· A TERM RELATION R IS <u>ADEQUATE</u> IFF

· $\vdash^{\wedge} e \, R \, f : \mathrm{bool} \Rightarrow [\![e]\!]_{\varepsilon} = [\![f]\!]_{\varepsilon}$

· A TERM RELATION R IS PRE-ADEQUATE IFF

$$\cdot \vdash e \overset{\Omega}{R} f : \text{bool} \implies [\![e]\!]_\xi \leq [\![f]\!]_\xi$$

(I.E. IF $[\![e]\!]_\xi = \text{true}$ THEN $[\![f]\!]_\xi = \text{true}$ AND IF $[\![e]\!]_\xi = \text{false}$ THEN $[\![f]\!]_\xi = \text{false}$)

$$\Gamma \vdash e \overset{\text{CTX}}{\simeq} f : \sigma \iff \forall C[\Gamma \vdash \sigma] : \text{bool}$$

$$[\![C[e]]\!]_\xi = [\![C[f]]\!]_\xi$$

PROPOSITION

- THERE IS A LARGEST COMPATIBLE AND ADEQUATE TERM RELATION $\overset{\text{CTX}}{\simeq}$, CALLED CONTEXTUAL EQ
- THERE IS A LARGEST COMPATIBLE AND PRE-ADEQUATE TERM RELATION $\overset{\text{CTS}}{\lesssim}$, CALLED THE CONTEXTUAL PREORDER

DENOTATIONAL EQUIVALENCES

$$[\![\text{unit}]\!]_\omega = \{*\} \qquad [\![\text{bool}]\!]_\omega = \{\text{false}, \text{true}\}$$

$$[\![\sigma \times \tau]\!]_\omega = [\![\sigma]\!]_\omega \times [\![\tau]\!]_\omega \quad \dots$$

$$x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash e : \tau$$

$$[\![e]\!]_\omega \in [\![\sigma_1]\!] \times \dots \times [\![\sigma_n]\!] \to [\![\tau]\!]$$

· THIS ALLOWS US TO DEFINE A

TERM RELATION $\overset{D}{\simeq}$ SUCH THAT

$$\frac{[\![e]\!]_D = [\![f]\!]_D}{\Gamma \vdash e \overset{D}{\simeq} f : \sigma}$$

- THEOREM $\overset{D}{\simeq} = \overset{CTX}{\simeq}$

LOGICAL RELATIONS

- ONE CAN DEFINE A CLOSED TERM RELATION $\overset{L}{\simeq}$ CALLED LOGICAL EQUIVALENCE AS FOLLOWS

  - $\vdash^{v} v \overset{L}{\simeq} w : bool \iff v = w$
  - $\vdash^{v} v \overset{L}{\simeq} w : unit$
  - $\vdash^{v} \langle v, w \rangle \overset{L}{\simeq} \langle u, z \rangle : \sigma \times \tau \iff \vdash^{v} v \overset{L}{\simeq} u : \sigma$ AND $\vdash^{v} w \overset{L}{\simeq} z : \tau$
  - $\vdash^{v} \lambda x. e \overset{L}{\simeq} \lambda x. f : \sigma \to \tau \iff$

    $$\forall v, w \left(\cdot \vdash v \overset{L}{\simeq} w : \sigma \implies \cdot \vdash e[v/x] \overset{L}{\simeq} f[w/x] : \tau \right)$$

  - $\vdash^{\Lambda} e \overset{L}{\simeq} f : \sigma \iff \vdash [\![e]\!]_{\Sigma} \overset{L}{\simeq} [\![f]\!]_{\Sigma} : \sigma$

THEOREM

$\overset{L}{\simeq} = \overset{CTX}{\simeq}$

RECURSIVE TYPES & APPLICATIVE BISIMILARITY

$$\sigma ::= \mu t.\sigma \mid t \mid \sigma + \tau \mid \text{void}$$

$$v ::= \text{inl } v \mid \text{inr } v \mid \text{fold } v$$

$$e ::= \text{abort } v \mid \text{case } v \text{ of inl } x \to e \mid \text{inr } x \to f \mid \text{unfold } v$$

STATICS

$$\frac{\Gamma \vdash^{v} v : \sigma[\mu t.\sigma / t]}{\Gamma \vdash^{v} \text{fold } v : \mu t.\sigma}$$

$$\frac{\Gamma \vdash^{v} v : \mu t.\tau}{\Gamma \vdash^{\Lambda} \text{unfold } v : \sigma[\mu t.\sigma / t]}$$

DYNAMICS

$$[\![\text{unfold}(\text{fold}(v))]\!]^{n+2}_{\xi} = v$$

$$[\![\text{case inl } v \text{ of } (\text{inl } x \mapsto e \mid \text{inr } x \mapsto f)]\!]^{n+1}_{\xi} = [\![e[v/x]]\!]^{n}_{\xi}$$

$\{ [\![-]\!]^{n}_{\xi} \}_{n \in \mathbb{N}}$ IS AN ω-CHAIN

- DEFINING LOGICAL EQUIVALENCE AND DENOTATIONAL EQUIVALENCE TO THIS CALCULUS IS POSSIBLE BUT NONTRIVIAL
  - L.E. $\longmapsto$ STEP INDEXING
  - D.E. $\longmapsto$ DOMAIN THEORY
- THE OBTAINED CALCULUS, Λ^{μ} CAN BE ON THE OTHER HAND, EASILY ENDOWED WITH CONTEXTUAL EQUIVALENCE.
- TOMORROW WE WILL INTRODUCE A DIFFERENT NOTION OF EQUIVALENCE FOR Λ^{μ}, CALLED APPLICATIVE BISIMILARITY.