FROM PROGRAM EQUIVALENCES TO PROGRAM METRICS

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OPLSS

Λ^{ST} $\quad \overset{CTX}{\simeq} = \overset{D}{\simeq} = \overset{L}{\simeq}$ $\qquad \Lambda^{\mu}$ $\quad \overset{CTX}{\simeq} = \overset{a}{\simeq}$

APPLICATIVE BISIMILARITY

A CTR R IS AN APPLICATIVE SIMULATION IF THE FOLLOWING CLAUSES ARE SATISFIED

- $\vdash^{v} \langle v, w\rangle \, R \, \langle u, z\rangle : \sigma \times \tau \Rightarrow$
  $$\vdash^{v} v \, R \, u : \sigma \quad \wedge \quad \vdash^{v} w \, R \, z : \tau$$
- $\vdash^{v} v \, R \, w : \sigma + \tau \Rightarrow$
  $$\begin{cases} v = \mathrm{inl}\ u \Rightarrow w = \mathrm{inl}\ z \wedge \vdash^{v} u \, R \, z : \sigma \\ v = \mathrm{inr}\ u \Rightarrow w = \mathrm{inr}\ z \wedge \vdash^{v} u \, R \, z : \tau \end{cases}$$
- $\vdash^{v} \mathrm{fold}\, v \, R \, \mathrm{fold}\, w : \mu t.\sigma \Rightarrow$
  $$\vdash^{v} v \, R \, w : \sigma[t/\mu t.\sigma]$$
- $\vdash^{v} \lambda x.e \, R \, \lambda x.f : \sigma \to \tau \Rightarrow$
  $$\forall v : \mathcal{V}^{\bullet}_{\sigma} . \vdash e[v/x] \, R \, f[v/x] : \tau$$
- $\vdash^{\Lambda} e \, R \, f : \sigma \Rightarrow$
  $$[\![e]\!]_{\varepsilon} = v \Rightarrow [\![f]\!] = w \wedge \vdash^{v} v \, R \, w : \sigma$$

A WAY TO TURN A RELATION $R : X \nrightarrow Y$ INTO $\tilde{\mathcal{M}}R : X_{\perp} \nrightarrow Y_{\perp}$

$[\![e]\!]_{\varepsilon} \, \tilde{\mathcal{M}}R^{v}_{\sigma} \, [\![f]\!]_{\varepsilon}$

- APPLICATIVE SIMILARITY : $\overset{a}{\lesssim}$
  - THE UNION OF ALL APPLICATIVE SIMULATIONS

- APPLICATIVE BISIMILARITY: $\overset{e}{\simeq} = \overset{e}{\lesssim} \cap (\overset{e}{\lesssim})^T$

- AN ALTERNATIVE DEFINITION:

$$R \xrightarrow{[\cdot]} [R]$$

$$[R]^{v}_{unit} = I^{v}_{unit}$$

$$\vdots$$

$$[R]^{\Lambda}_{\sigma} = [\![-]\!]_{\varepsilon} ; \tilde{\mathcal{M}} R^{v}_{\sigma} ; [\![-]\!]_{\varepsilon}^{T}$$

APPLICATIVE BISIMILARITY $\nu R.[R]$

- YOU CAN PROVE THAT $e \overset{CTX}{\simeq} f$ BY JUST SHOWING THAT THERE IS ONE APPLIC. BISIMULATION R SUCH THAT eRf:

$$eRf \Rightarrow e \overset{e}{\simeq} f \Rightarrow e \overset{CTX}{\simeq} f$$

THIS IS DONE BY SO-CALLED HOWE'S TECHNIQUE

$$R \overset{?}{\underset{\subseteq}{\longmapsto}} R^H = \mu X. \hat{X} ; R^\circ$$

YOU ARE SURE, BY CONSTRUCTION, THAT R^H IS A CONGRUENCE WITH MINIMAL CONDITIONS.

LEMMA. IF R IS A REFLEXIVE AND TRANSITIVE APPLICATIVE SIMULATION, THEN $(R^H)^C \subseteq [(R^H)^C]$

COROLLARY $\lesssim^{a} \subseteq \lesssim^{CTX}$ $\quad \simeq^{a} \subseteq \simeq^{CTX}$

EXERCISE $\lambda x.x \simeq^{CTX} \lambda x.Ix$

PROBABILISTIC EFFECTS

$e ::= \ldots \mid \text{sample}_q$ $\qquad \Gamma \vdash \text{sample}_q : \text{bool}$

$q \in \mathbb{Q}_{(0,1)}$ $\qquad M \oplus_q N$

DYNAMICS

DISTRIBUTIONS

$\mathcal{D} : X \longrightarrow \mathbb{R}_{[0,1]}$ SUCH THAT

- $\mathcal{D}(x) > 0$ FOR DENUMERABLY MANY $x \in X$

- $\sum_{x \in X} \mathcal{D}(x) = 1$ PROPER DISTRIBUTIONS $\qquad \sum_{x \in X} \mathcal{D}(x) \leq 1$ SUBDISTRIBUTIONS

$\emptyset \longrightarrow$ EMPTY SUBDISTRIBUTION

$\emptyset(v) = 0$

$\delta_v \longrightarrow$ DIRAC (SUB)DISTRIBUTION

$\delta_v(w) = \{ 1 \quad \text{IF } v = w$

$$\begin{cases} \ \\ 0 & \text{OTHERWISE} \end{cases}$$

$$[\![e]\!]^0_\varepsilon = \emptyset$$

$$[\![\mathrm{sample}_q]\!]^{n+1} = \overbrace{q \cdot \delta_{\mathrm{true}} + (1-q) \cdot \delta_{\mathrm{false}}}^{\text{CONVEX COMBINATION}}$$

$$[\![\mathrm{return}\ v]\!]^{n+1} = \delta_v$$

$$\vdots$$

$$[\![\mathrm{let}\ x = e\ \mathrm{in}\ f]\!]^{n+1} = \sum_v [\![e]\!]^n(v) \cdot [\![f[v/x]]\!]^n$$

$$[\![-]\!] : \Lambda^{\bullet}_{\sigma} \longrightarrow \mathbf{D}(\mathcal{V}^{\bullet}_{\sigma})$$

LOGICAL RELATIONS

(FOR THE SAKE OF SIMPLICITY, LET'S NOT CONSIDER FIXPOINTS)

IN THE DETERMINISTIC CASE

$$\overset{L}{\underset{\sigma}{\approx}}{}^{\Lambda} = [\![-]\!]_{\varepsilon} ; \underbrace{\overset{L}{\underset{\sigma}{\approx}}{}^{\mathcal{V}}} ; [\![-]\!]^{T}_{\varepsilon}$$

$e \quad f$

IN THE PROBABILISTIC CASE

$$\overset{L}{\underset{\sigma}{\approx}}{}^{\Lambda} = [\![-]\!]_{\varepsilon} ; \underbrace{\widetilde{\mathcal{U}}\ \overset{L}{\underset{\sigma}{\approx}}{}^{\mathcal{V}}} ; [\![-]\!]^{+}_{\varepsilon}$$

ONE NEEDS, IN OTHER WORDS, A WAY TO "LIFT" $\overset{L}{\underset{\sigma}{\approx}}{}^{\mathcal{V}}$ TO A RELATION ON DISTRIBUTIONS OF VALUES.

- GIVEN TWO DISTRIBUTIONS $\mathcal{D}$ AND $\mathcal{E}$ OVER X AND Y, WE SAY THAT A THIRD DISTRIBUTION $\mathcal{C}$ OVER $X \times Y$ IS A COUPLING WHEN

$$\sum_{y} \mathcal{C}(x,y) = \mathcal{D}(x)$$

$$\sum_{x} \mathcal{C}(x,y) = \mathcal{E}(y)$$

- GIVEN A RELATION $R : X \nrightarrow Y$, THE RELATION $\tilde{\mathcal{D}}R : \mathbf{D}X \nrightarrow \mathbf{D}Y$ IS DEFINED AS FOLLOWS:

$$\mathcal{D}\,(\tilde{\mathcal{D}}R)\,\mathcal{E} \iff \exists\, \mathcal{C} \in \overbrace{\Omega(\mathcal{D},\mathcal{E})}^{\text{THE SET OF ALL COUPLINGS}}.$$

$$\mathcal{C}(x,y) > 0 \implies x R y$$

- <u>APPLICATIVE BISIMILARITY</u>

$$R \xrightarrow{[\cdot]} [R]$$

$$[R]^{\wedge}_{\sigma} = [\![-]\!]_{\varepsilon} \,;\, \underbrace{\tilde{\mathcal{D}}\underbrace{\tilde{\mathcal{M}}R^{\vee}_{\sigma}}_{X_\perp \nrightarrow X_\perp}}_{\underline{\mathbf{D}}X_\perp \nrightarrow \mathbf{D}X_\perp} \,;\, [\![-]\!]^{\dagger}_{\varepsilon}$$

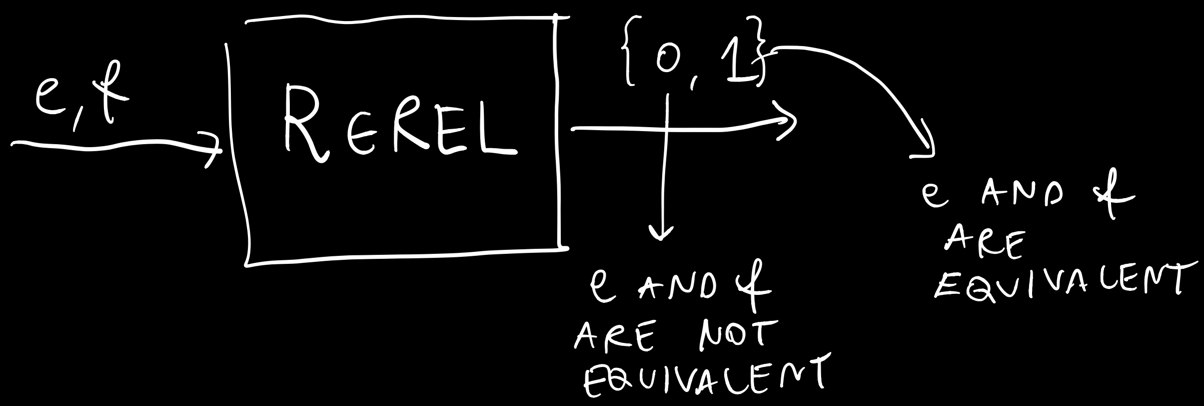
A SUBDISTRIBUTION ON X CAN BE SEEN AS A DISTRIBUTION OVER $X_\perp$

UR.[R] $\longrightarrow$ $\simeq^{R} = \simeq^{CTX}$

- THIS SCALES TO CALCULI WITH ALGEBRAIC EFFECTS (NONDETERMINISM, GLOBAL STORE, EXCEPTIONS, I/O)

PROGRAM METRICS

- UP TO NOW, PROGRAMS HAVE BEEN COMPARED BY WAY OF RELATIONS



EQUIVALENCE	PSEUDOMETRICS
$R(x,x)$	$\mu(x,x) \leq 0$
$R(x,y) \Rightarrow R(y,x)$	$\mu(x,y) \geq \mu(y,x)$
$R(x,y) \wedge R(y,z) \Rightarrow R(x,z)$	$\mu(x,y) + \mu(y,z) \geq \mu(x,z)$

HOW ABOUT COMPATIBILITY?

$$e \mathrel{R} f \Rightarrow \forall C.\ C[e] \mathrel{R} C[f]$$

μ?

$$\mu(e,f) \geq \sup_{C} \mu(C[e], C[f])$$