Kleene Algebras and Applications

Alexandra Silva

Course Content

Lecture 1

Kleene algebra
Brzozowski derivatives
Antimirov derivatives
Equivalence via automata

Lecture 2

Coinduction up-to

Lecture 3

Equivalence via axioms
Completeness

Lecture 4

Extensions with tests, observations, and concurrency

Lecture 1

Context

```
while a & b do
   p;
od;
while a do
   q ;
   while a & b do
      p;
   od
od
```

$\stackrel{?}{\equiv}$

```
while a do
   if b then
      p;
   else
   q;
od
```

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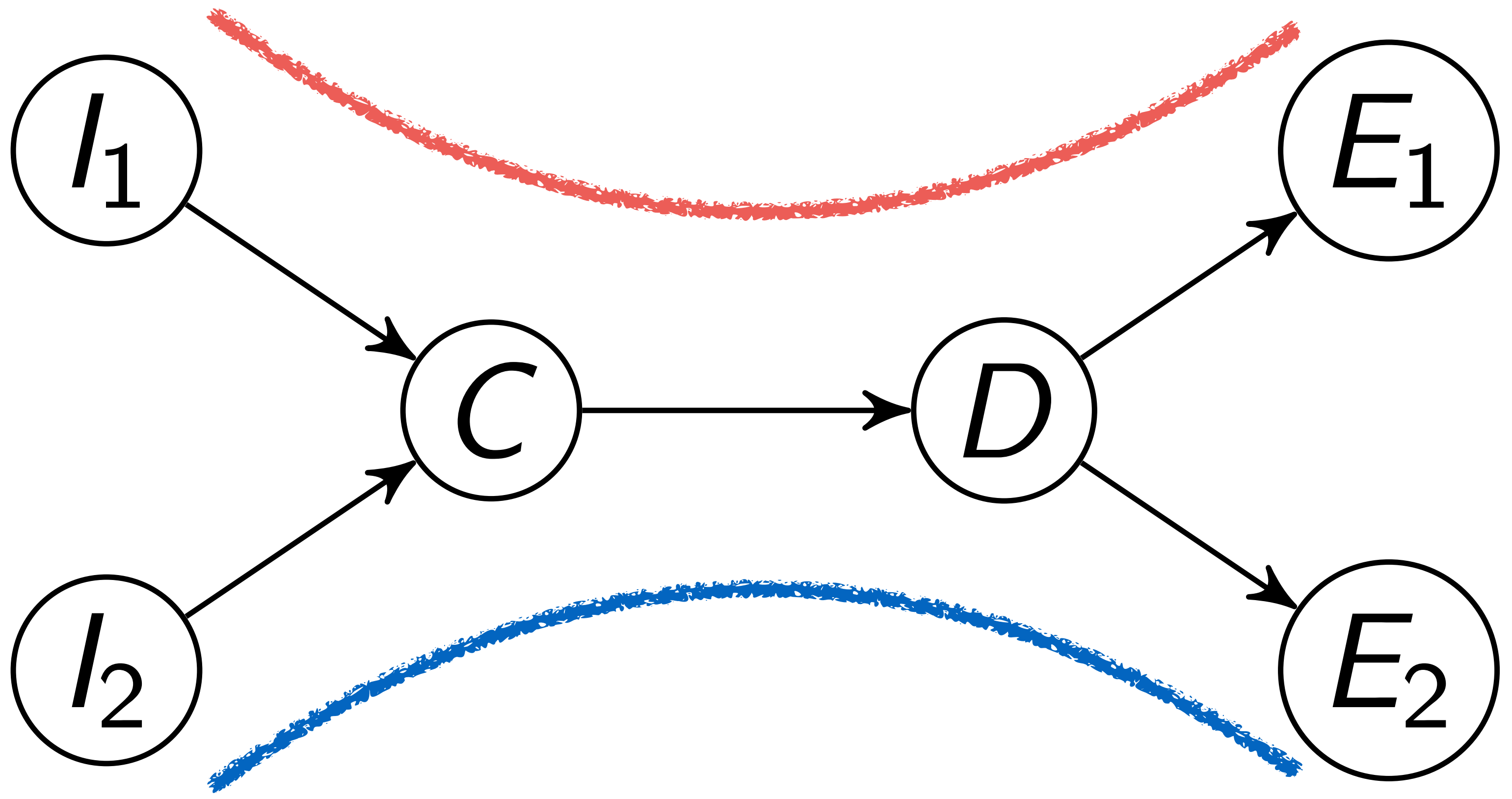
$$I_1 \; ; p \; ; E_1 \equiv 0$$



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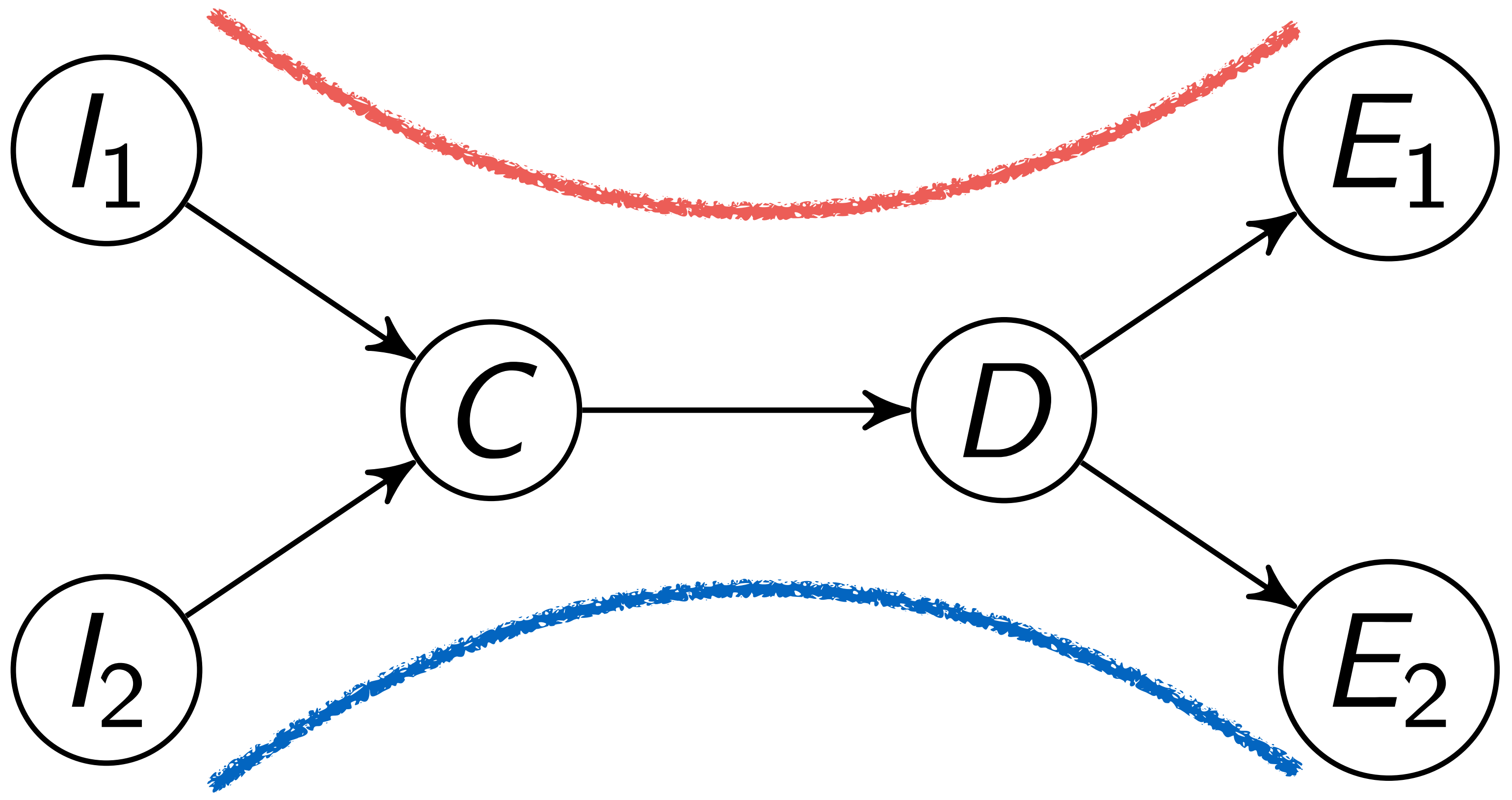
```

Verification via Program Equivalence

$$I_1 \; ; p \; ; E_1 \equiv 0$$

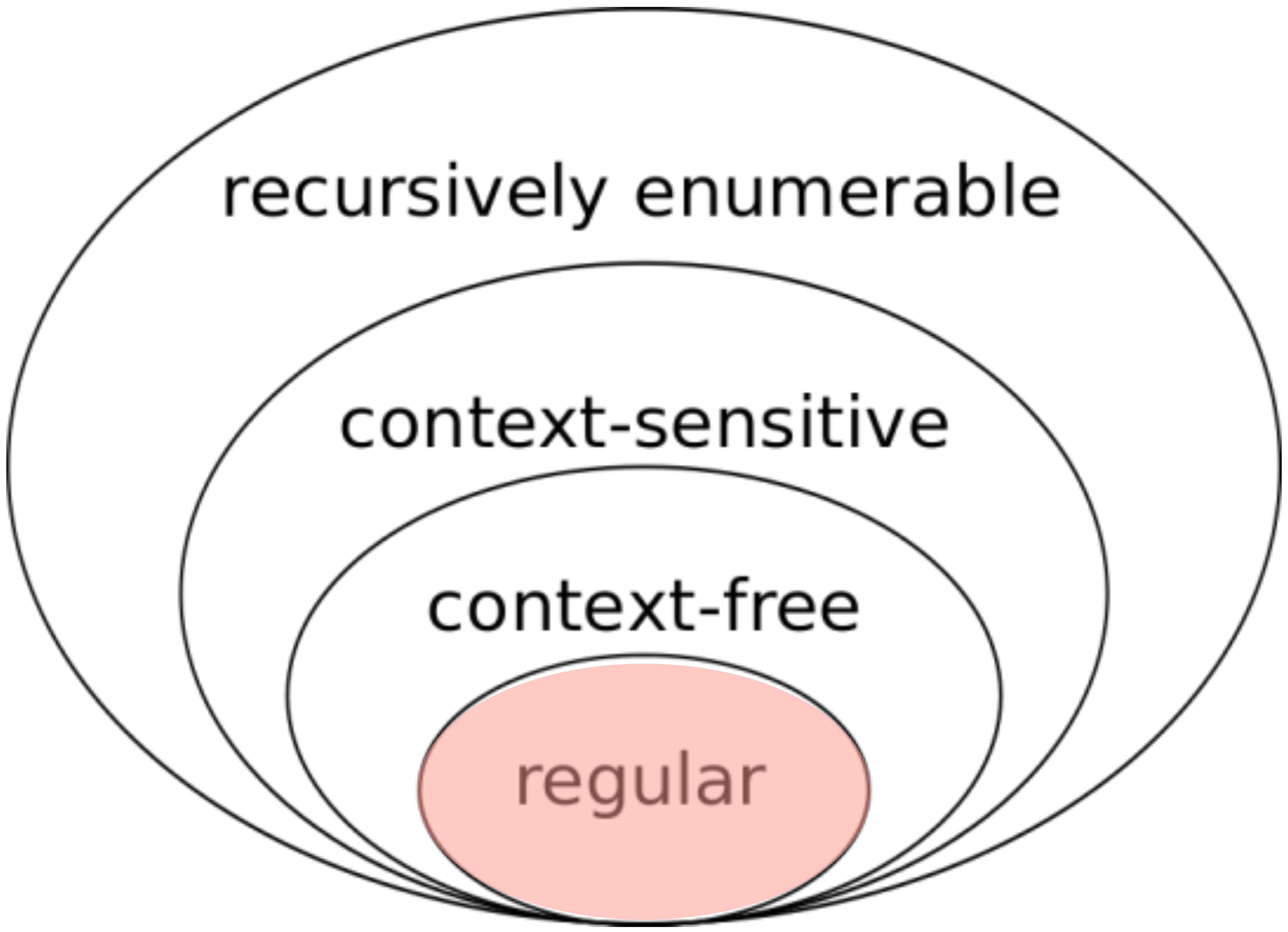


Languages of traces

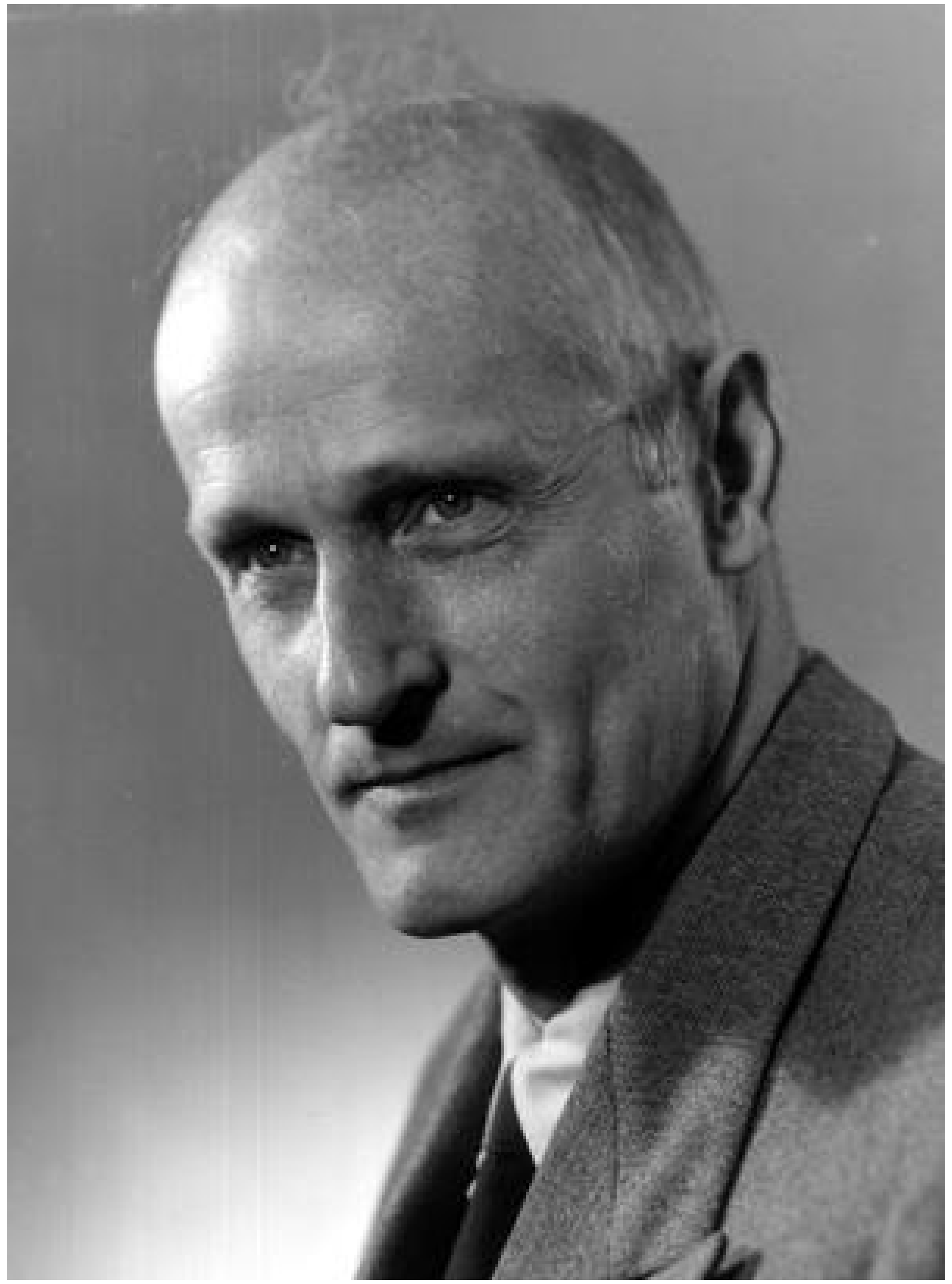
Some notation

A - finite alphabet
A* - finite words over A
Empty word - ε
Language - subset of words
$L \subseteq A^*$

Chomsky hierarchy



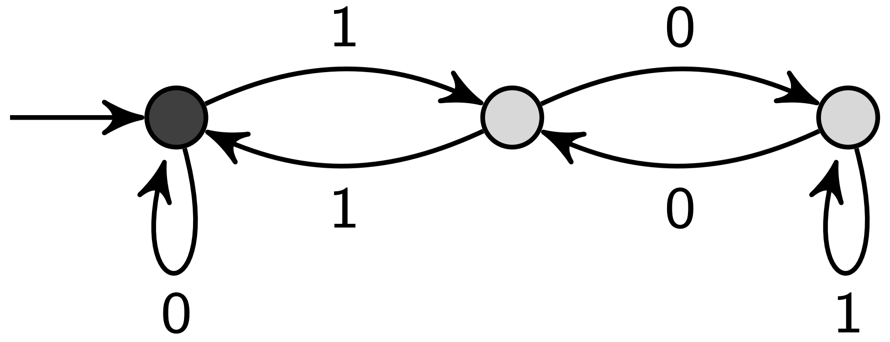
Kleene Algebra



Stephen Cole Kleene
(1909–1994)

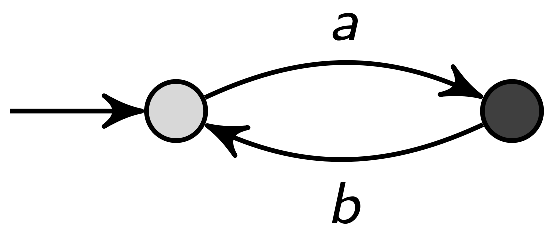
$(0 + 1(01^*0)^*1)^*$

{multiples of 3 in binary}



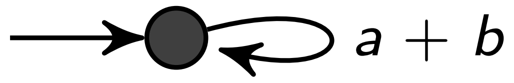
$(ab)^*a = a(ba)^*$

$\{a, aba, ababa, \ldots\}$



$(a + b)^* = a^*(ba^*)^*$

{all strings over $\{a, b\}$}



Regular expressions

$$r, r_1, r_2 ::= \underline{1} \mid \underline{0} \mid a \in A \mid r_1 + r_2 \mid r_1 r_2 \mid r^*$$

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Language Semantics

$$L(\underline{1}) = \{\epsilon\} \qquad L(\underline{0}) = \emptyset \qquad L(a) = \{a\}$$

$$L(r_1 + r_2) = L(r_1) \cup L(r_2) \qquad L(r_1 r_2) = L(r_1) \cdot L(r_2) \qquad L(r^*) = L(r)^*$$

Exercise

What languages do these expressions describe?

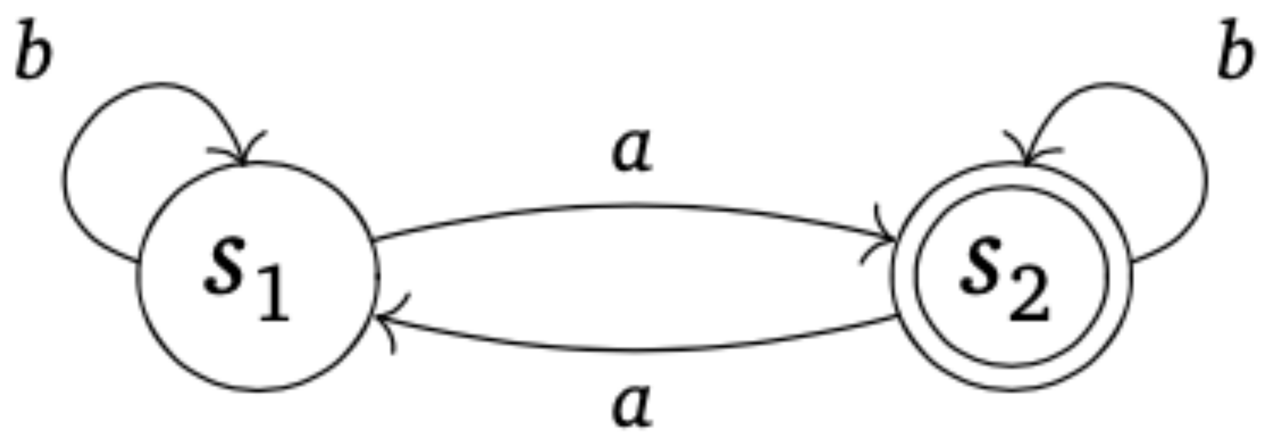
$$(a+b)(a+b)(a+b)^*$$

$$a^*ba + b^*ab$$

Give a regular expression describing this language?

$$L = \left\{ w \in A^* \,\middle|\, w \text{ contains } aba \text{ \underline{at least} once} \right\}$$

Deterministic Finite Automata



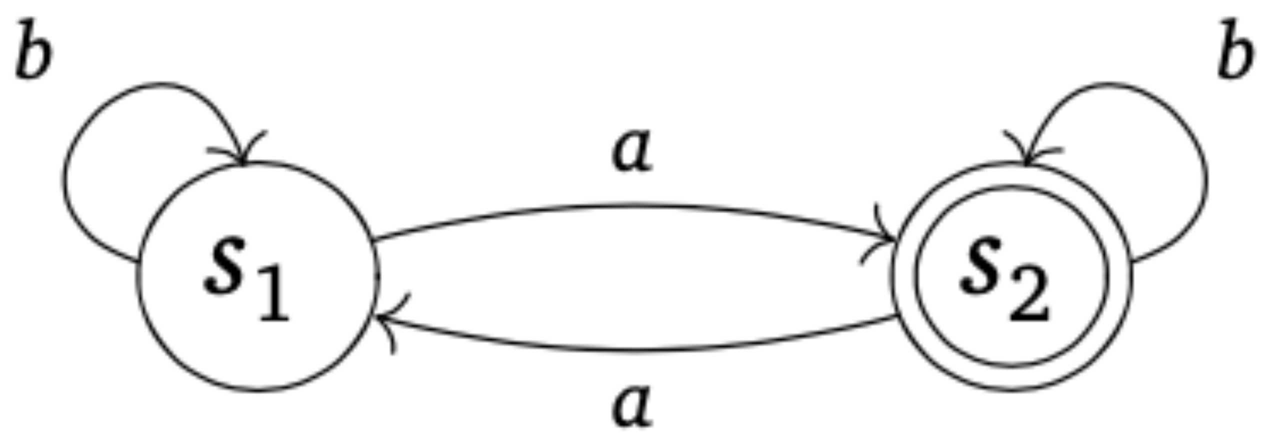
$$(S, \langle o, t \rangle)$$

States (set)

$$o \colon S \to 2$$ Final states

$$t \colon S \to S^A$$ Transition Function

Deterministic Finite Automata



$$(S, \langle o, t \rangle) \quad o \colon S \to 2 \quad t \colon S \to S^A$$

States (set)

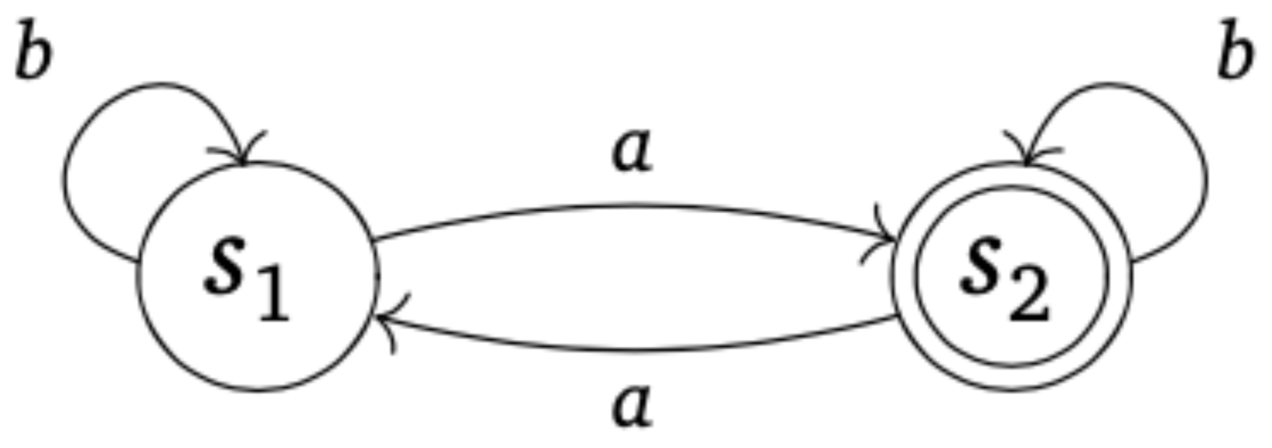
Final states

Transition Function

Inductive extension

$$t^*(s)(\varepsilon) = s$$
$$t^*(s)(aw) = t^*(t(s)(a))(w)$$

Deterministic Finite Automata



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Notation

$$s_w = t^*(s)(w)$$

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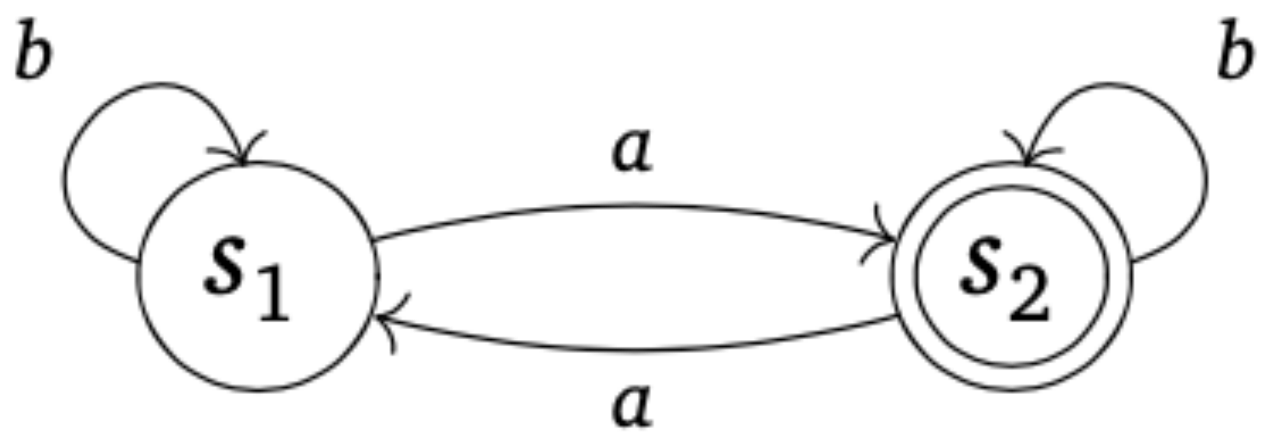

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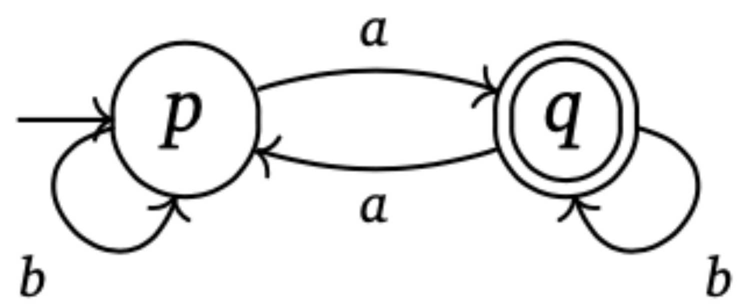
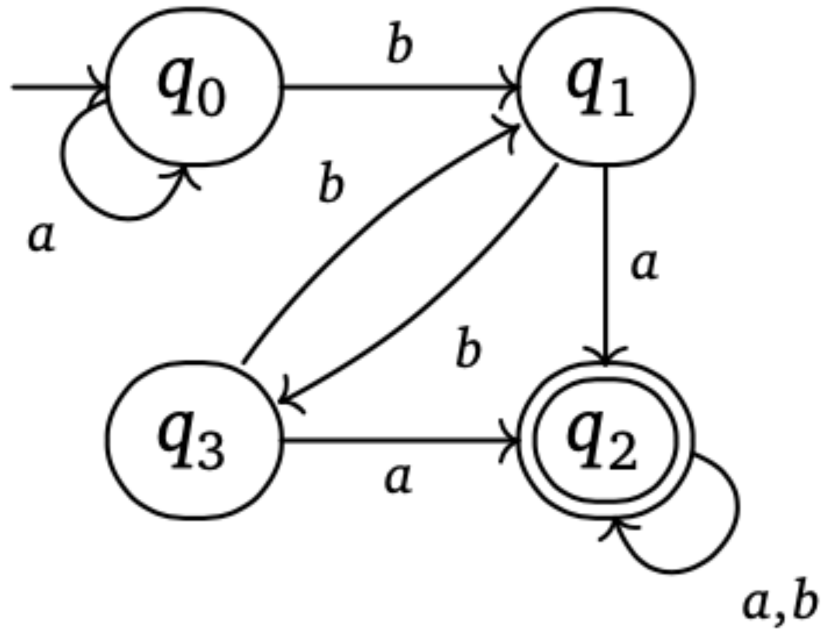
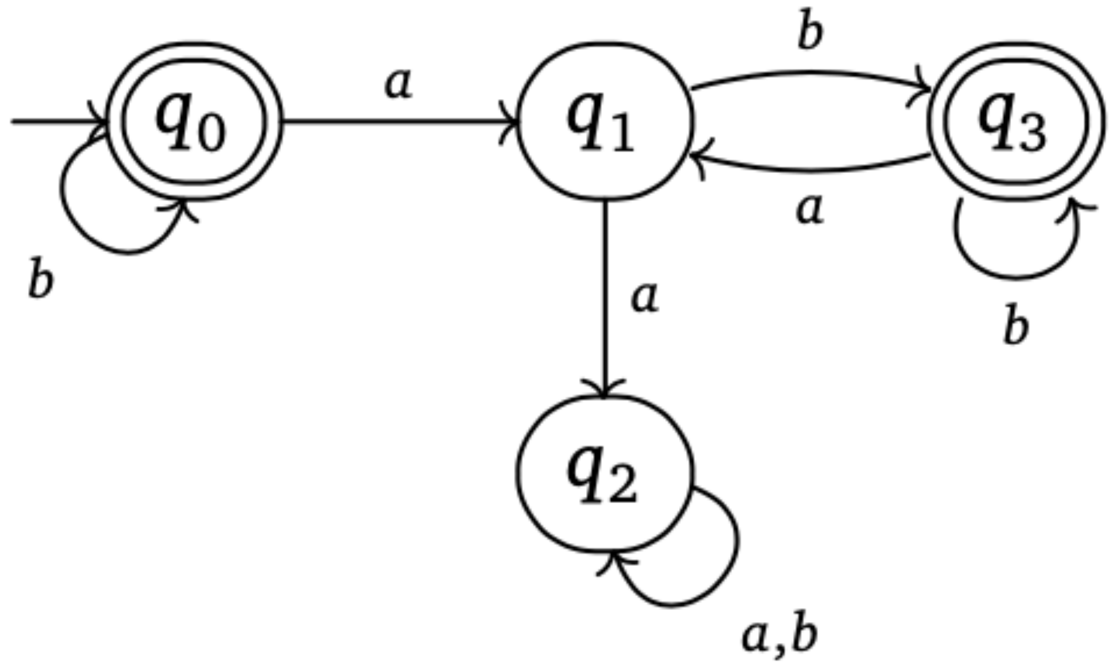
Language accepted by a state

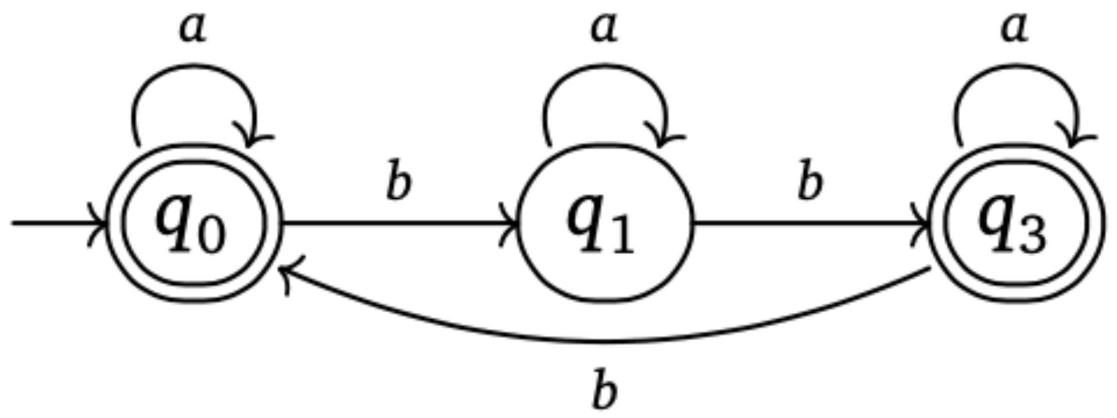
$$w \in L(s) \iff o(s_w) = 1$$

Exercise - DFA









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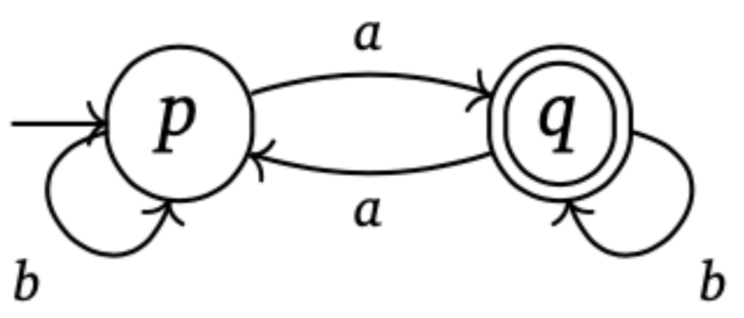
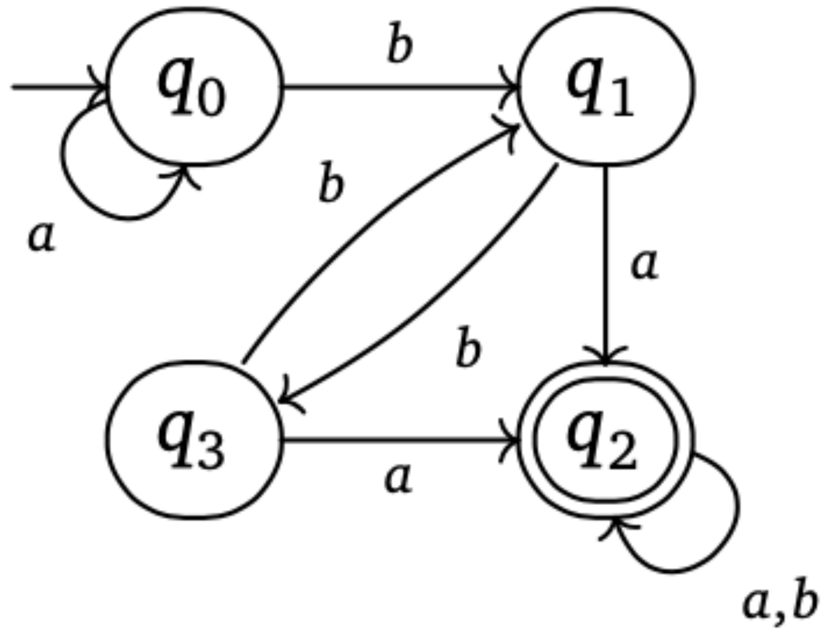
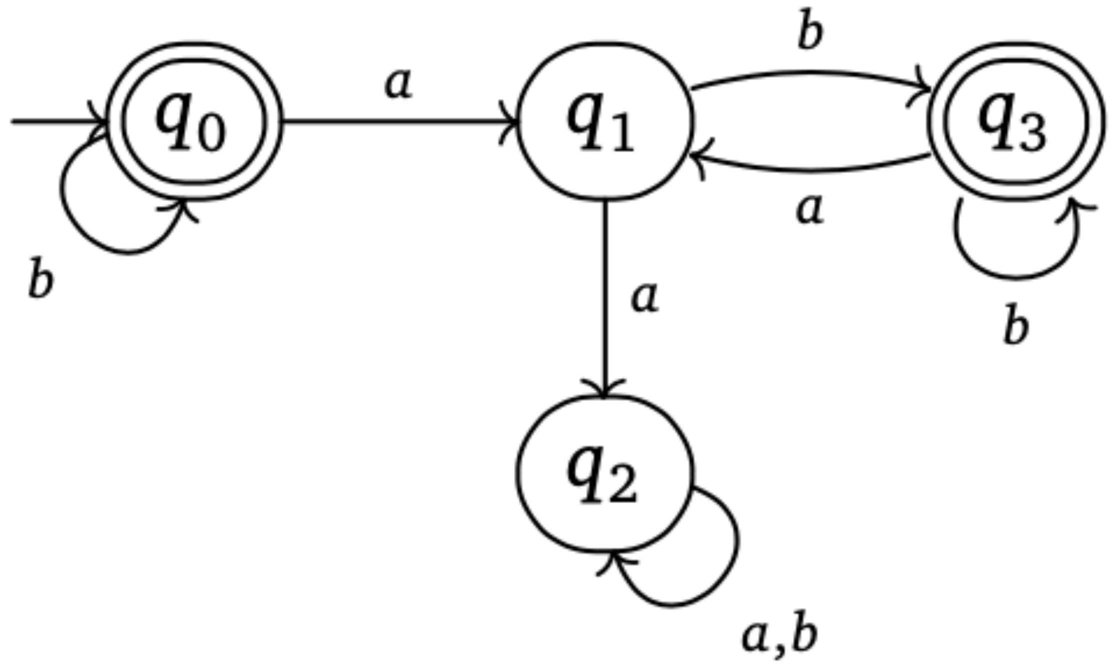


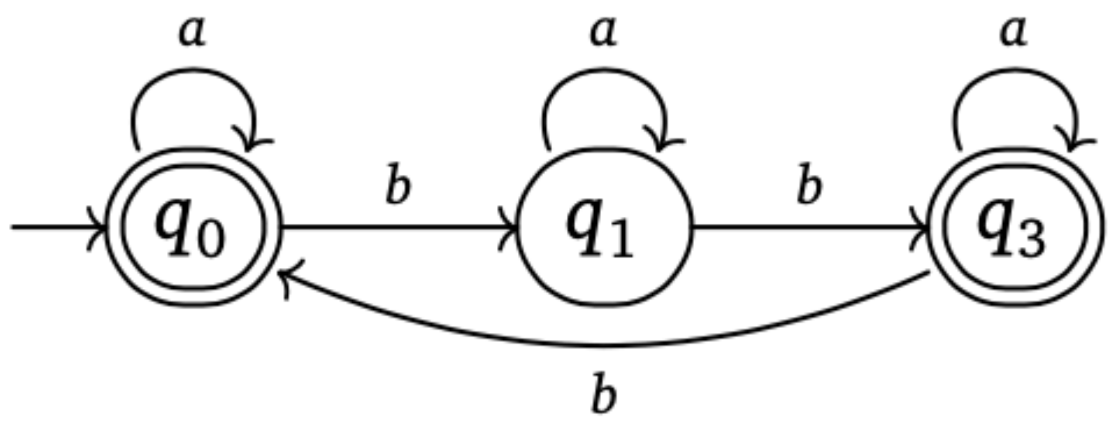
$\mathcal{L}(p) = \{w \in A^* \mid |w|_a \text{ is odd}\}.$



$\mathcal{L}(q_0) = \{w \in A^* \mid w \text{ has the subword } ba\}.$



$\mathcal{L}(q_0) = \{w \in A^* \mid \text{every } a \text{ in } w \text{ is followed by a } b\}.$



$\mathcal{L}(q_0) = \{w \in A^* \mid |w|_b = 3n + 2 \text{ or } |w|_b = 3n, n \in \mathbb{N}\}.$

Intermezzo: final coalgebra

$$
\begin{array}{ccc}
S & \overset{L}{\dashrightarrow} & 2^{A^*} \\
\Big\downarrow{\scriptstyle \langle o_S, t_S \rangle} & & \Big\downarrow{\scriptstyle \langle o_L, t_L \rangle} \\
2 \times S^A & \underset{id \times L^A}{\dashrightarrow} & 2 \times (2^{A^*})^A
\end{array}
$$

Kleene Algebra

$$
\begin{array}{llllll}
e_1 + (e_2 + e_3) & = & (e_1 + e_2) + e_3 & & & \text{(associativity of +)} \\
e_1 + e_2 & = & e_2 + e_1 & & & \text{(commutativity of +)} \\
e + e & = & e & & & \text{(idempotency of +)} \\
e + 0 & = & e & & & \text{(0 is an identity of +)} \\
\\
e_1(e_2e_3) & = & (e_1e_2)e_3 & & & \text{(associativity of }\cdot\text{)} \\
e1 & = & e & = & 1e & \text{(1 is an identity of }\cdot\text{)} \\
e0 & = & 0 & = & 0e & \text{(0 is an annihilator of }\cdot\text{)} \\
\\
(e_2 + e_3)e_1 & = & e_2e_1 + e_3e_1 & & & \text{(right distributivity)} \\
e_1(e_2 + e_3) & = & e_1e_2 + e_1e_3 & & & \text{(left distributivity)} \\
e^*e + \lambda & = & e^* & & & \\
ee^* + \lambda & = & e^* & & &
\end{array}
$$

Exercise

1. $x^*x^* = x^*$
2. $x^*=x^{**}$
3. $(x+y)^* = (x^*y)^*x^*$ denesting
4. $x(yx)^* = (xy)^*x$ sliding
5. $xy=yz \Rightarrow x^*y = yz^*$

Kleene Algebra, examples

- Regular languages
  - **+ is union**
  - **; is pointwise concatenation**
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  - **+ and ; lifted to usual matrices ops**
  - *** iteratively from 2 x 2**

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} (a + bd^*c)^* & (a + bd^*c)^* bd^* \\ (d + ca^*b)^* ca^* & (d + ca^*b)^* \end{bmatrix}$$

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Important for relational verification

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Important for automata representations

- Square Matrices over a KA **K**
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Kleene's Theorem

Theorem (Kleene'52): Let L be a subset of A^*. TFAE:

1. L is regular
2. L is accepted by a deterministic finite automaton

Proof

1 => 2 : Syntactic Brzozowski derivatives

2 => 1 : State elimination

Brzozowski Derivatives

$$r, r_1, r_2 ::= \underline{1} \mid \underline{0} \mid a \in A \mid r_1 + r_2 \mid r_1 r_2 \mid r^*$$

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$$\begin{aligned} o_{\mathcal{R}}(\underline{0}) &= 0 \\ o_{\mathcal{R}}(\underline{1}) &= 1 \\ o_{\mathcal{R}}(a) &= 0 \end{aligned}$$

$$\begin{aligned} o_{\mathcal{R}}(r_1 + r_2) &= o_{\mathcal{R}}(r_1) \vee o_{\mathcal{R}}(r_2) \\ o_{\mathcal{R}}(r_1 r_2) &= o_{\mathcal{R}}(r_1) \wedge o_{\mathcal{R}}(r_2) \\ o_{\mathcal{R}}(r^*) &= 1 \end{aligned}$$

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$$\begin{aligned} (\underline{0})_a &= \underline{0} \\ (\underline{1})_a &= \underline{0} \\ (a)_{a'} &= \begin{cases} \underline{1} & \text{if } a = a' \\ \underline{0} & \text{if } a \neq a' \end{cases} \end{aligned}$$

$$\begin{aligned} (r_1 + r_2)_a &= (r_1)_a + (r_2)_a \\ (r_1 r_2)_a &= \begin{cases} (r_1)_a r_2 & \text{if } o_{\mathcal{R}}(r_1) = 0 \\ (r_1)_a r_2 + (r_2)_a & \text{otherwise} \end{cases} \\ (r^*)_a &= r_a r^* \end{aligned}$$

Are we done?

$$(a^*)^*$$

$$\begin{aligned}
((a^*)^*)_a &= (\underline{1}a^*)(a^*)^* \\
((\underline{1}a^*)(a^*)^*)_a &= (\underline{0}a^* + \underline{1}a^*)(a^*)^* + (\underline{1}a^*)(a^*)^* \\
((\underline{0}a^* + \underline{1}a^*)(a^*)^* + (\underline{1}a^*)(a^*)^*)_a &= ((\underline{0}a^* + \underline{0}a^* + \underline{1}a^*)(a^*)^* + (\underline{1}a^*)(a^*)^*) + ((\underline{1}a^*)(a^*)^*)_a \\
&\vdots
\end{aligned}$$

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NOT FINITE!

Are we done?

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&\vdots
\end{aligned}$$

NOT FINITE!

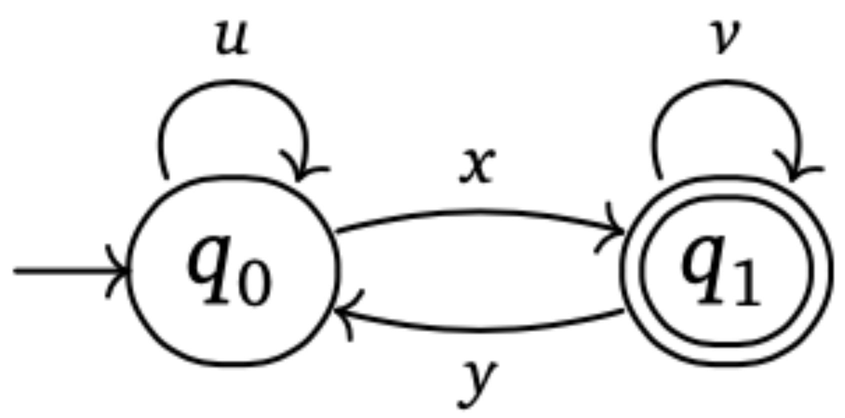
<u>Theorem</u> (Brzozowski): Let r be a regular expression. The set of syntactic Brzozowski derivatives is finite if it taken modulo ACI.

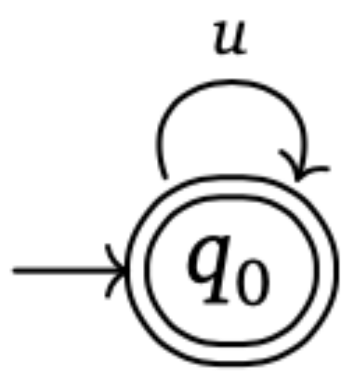
Intermezzo : final coalgebra

$$
\begin{array}{ccc}
\mathcal{R}(A) & \overset{L}{\dashrightarrow} & 2^{A^*} \\
\Big\downarrow {\scriptstyle \langle o_{\mathcal{R}}, t_{\mathcal{R}} \rangle} & & \Big\downarrow {\scriptstyle \langle o_L, t_L \rangle} \\
2 \times (\mathcal{R}(A))^A & \underset{id \times L^A}{\dashrightarrow} & 2 \times (2^{A^*})^A
\end{array}
$$

Alternative proof 1=>2
Thompson + epsilon-elim + subset construction

State Elimination





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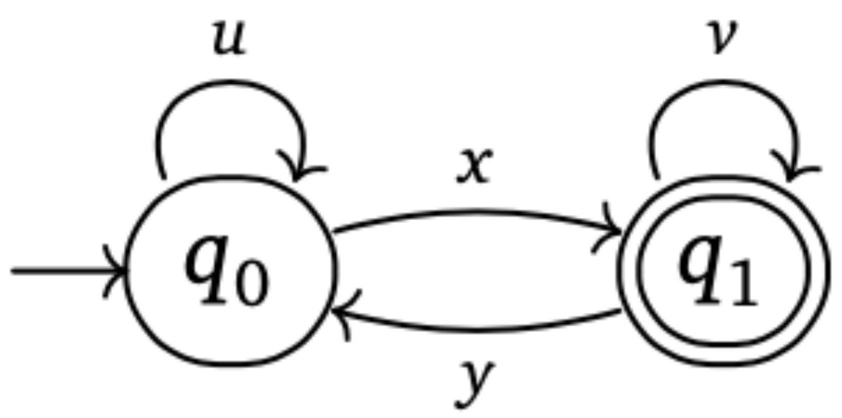
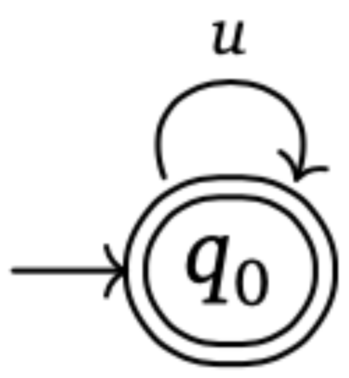
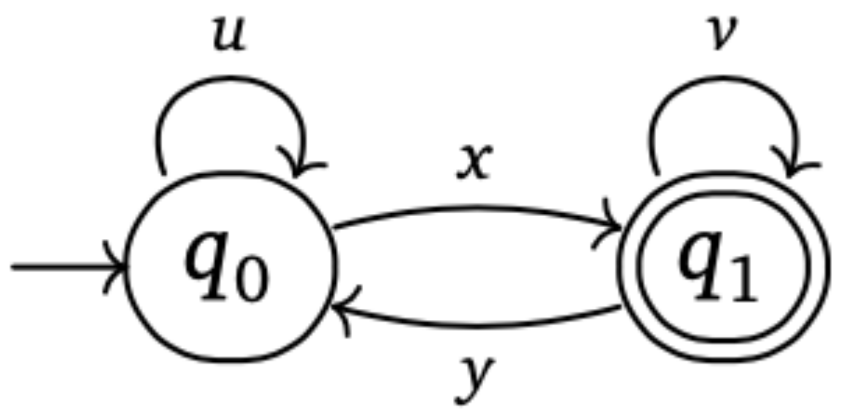
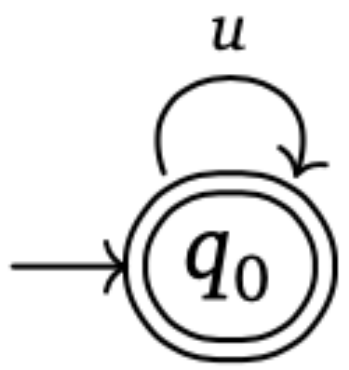




$$u^*x(v + yu^*x)^*$$

State Elimination





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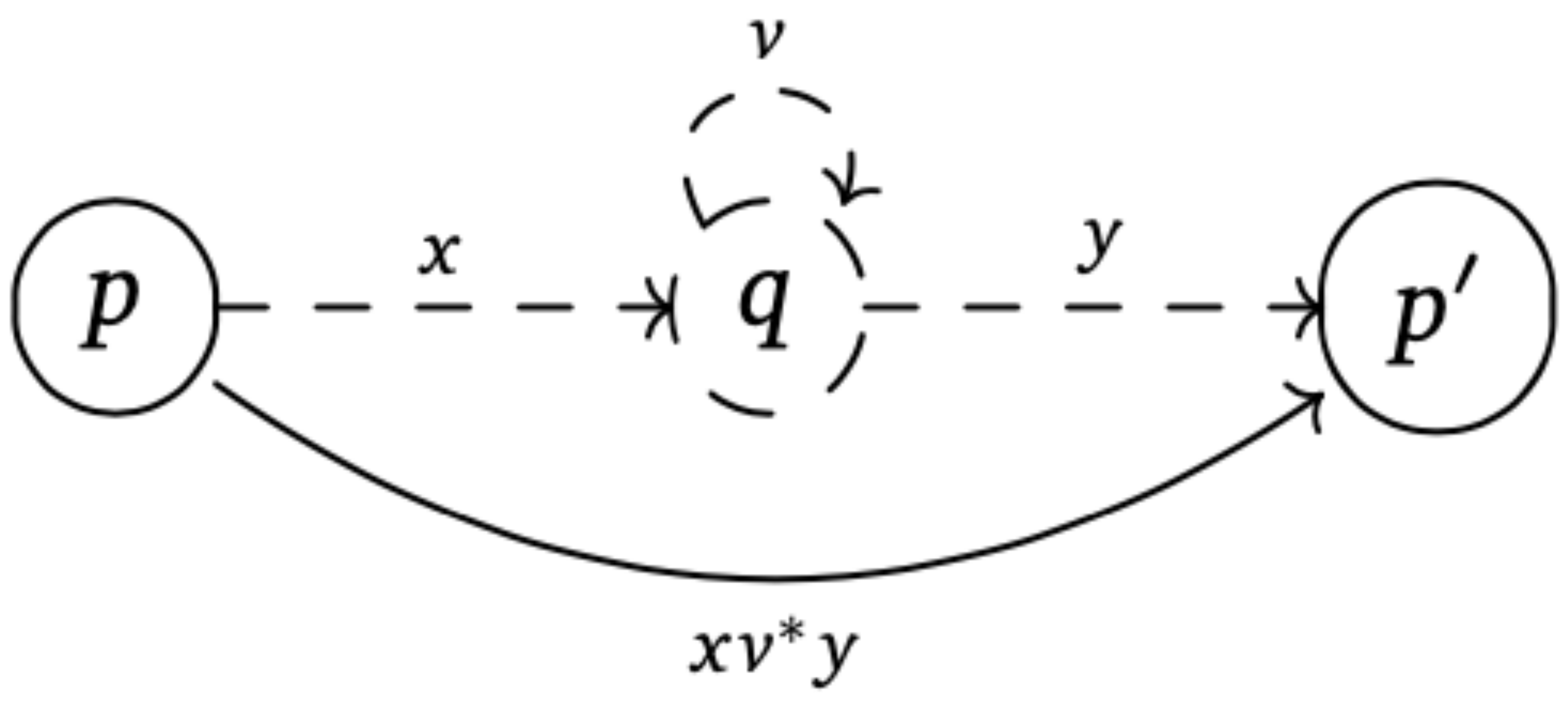
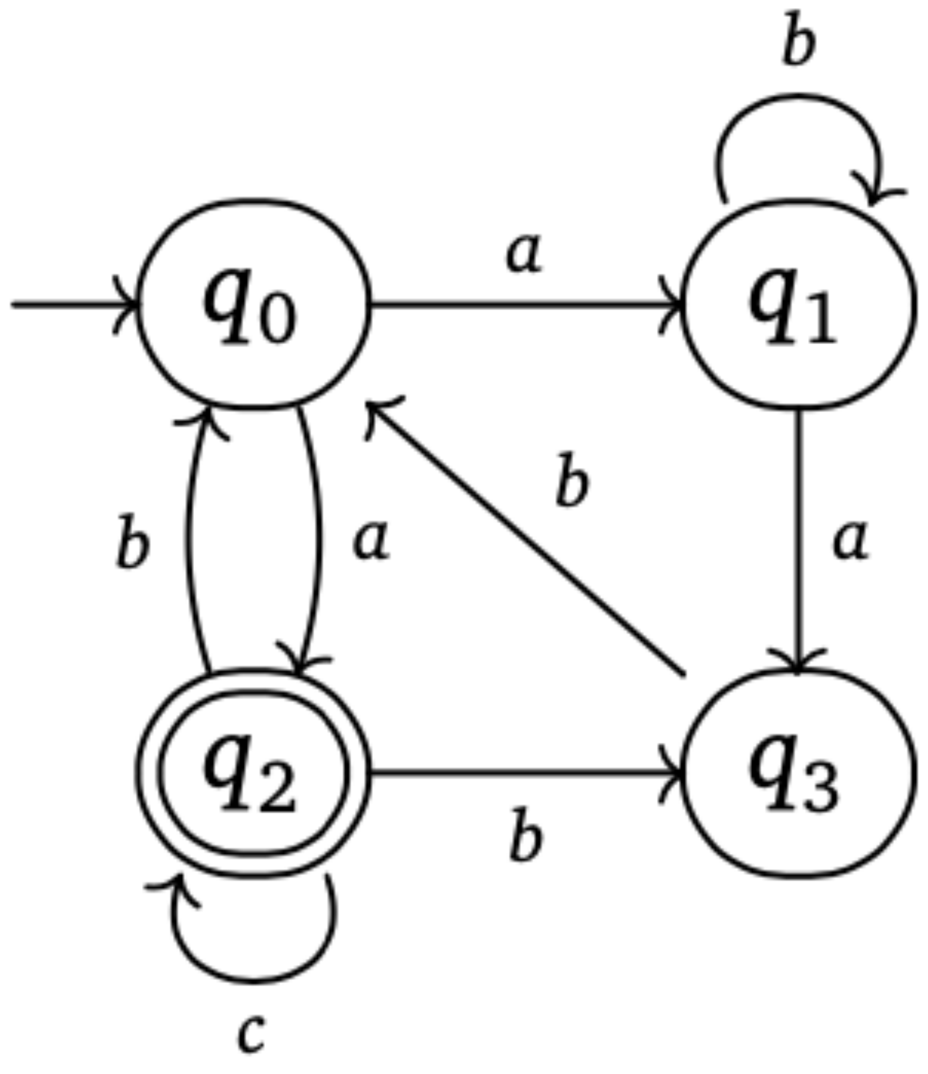
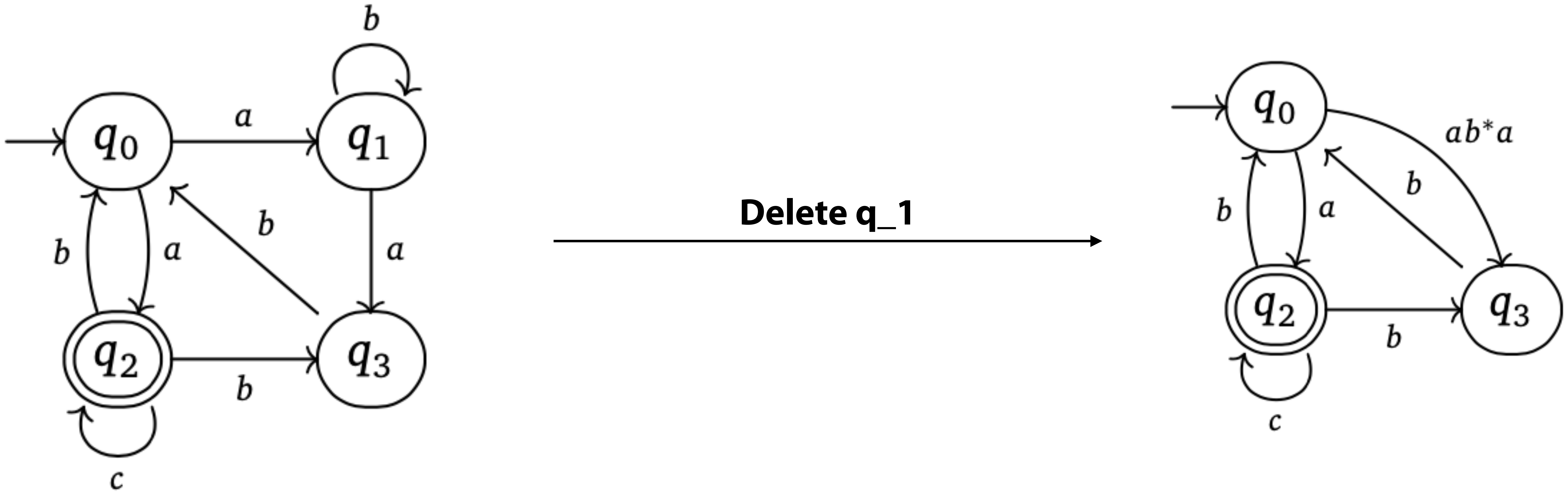


Exercise



Delete q_1

Exercise



Lecture 2

Alternative proof 2 => 1
Solving systems of equations

$$x \equiv r_1 + r_2 x$$

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Solving systems of equations

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$$r_s \equiv \sum_{a \in A} a r_{s_a} + \underline{o_S(s)}$$

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Solving systems of equations

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$$r_s \equiv \sum_{a \in A} a r_{s_a} + \underline{o_S(s)}$$

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NFAs

$$(a+b)^*a(a+b)(a+b)(a+b)(a+b)$$

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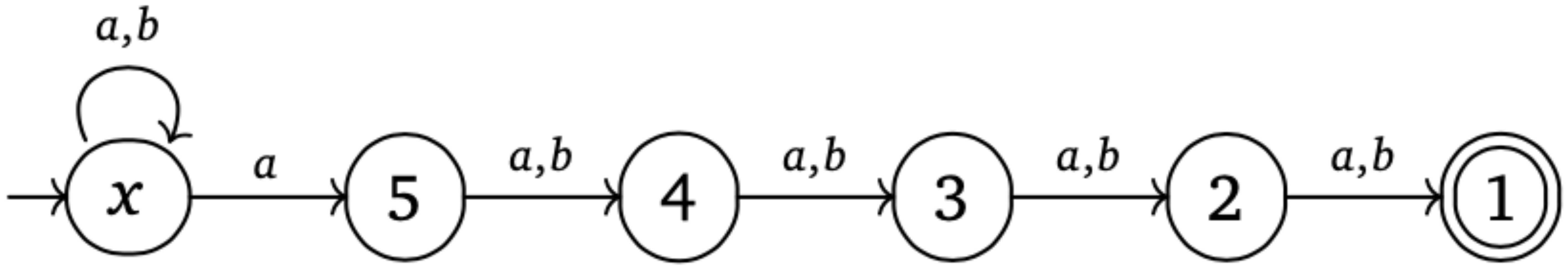


Antimirov Derivatives

Context

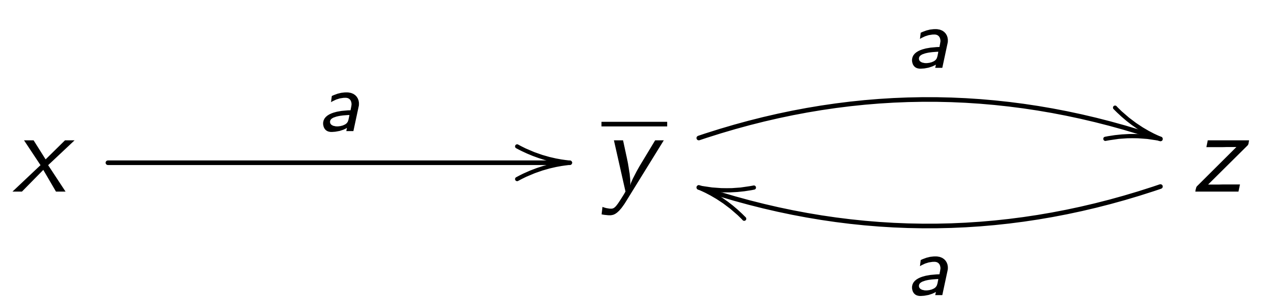
Tools and proof techniques for systems equivalence

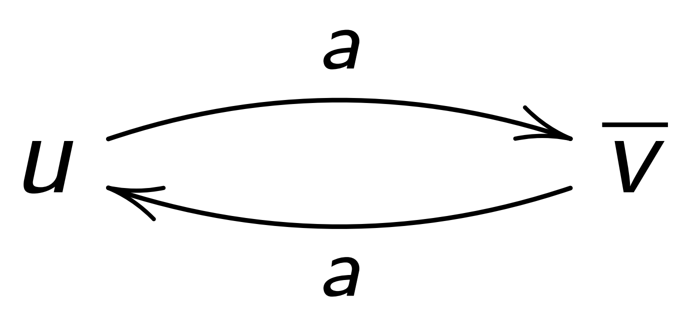
Methodology:

1. First do it naively
2. Then improve the associated proof method

Deterministic finite automata

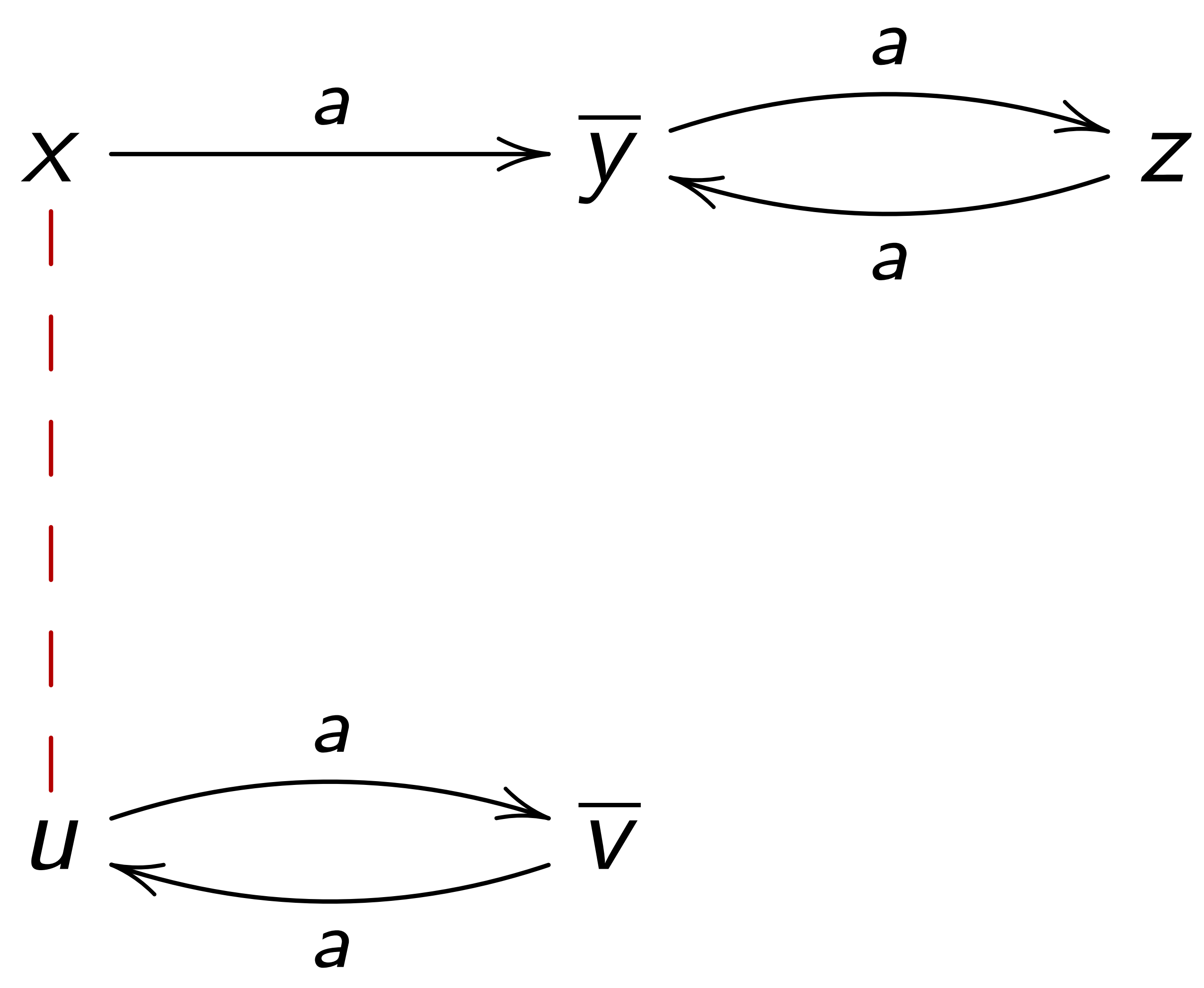
The states x and u are language equivalent





Deterministic finite automata

The states x and u are language equivalent



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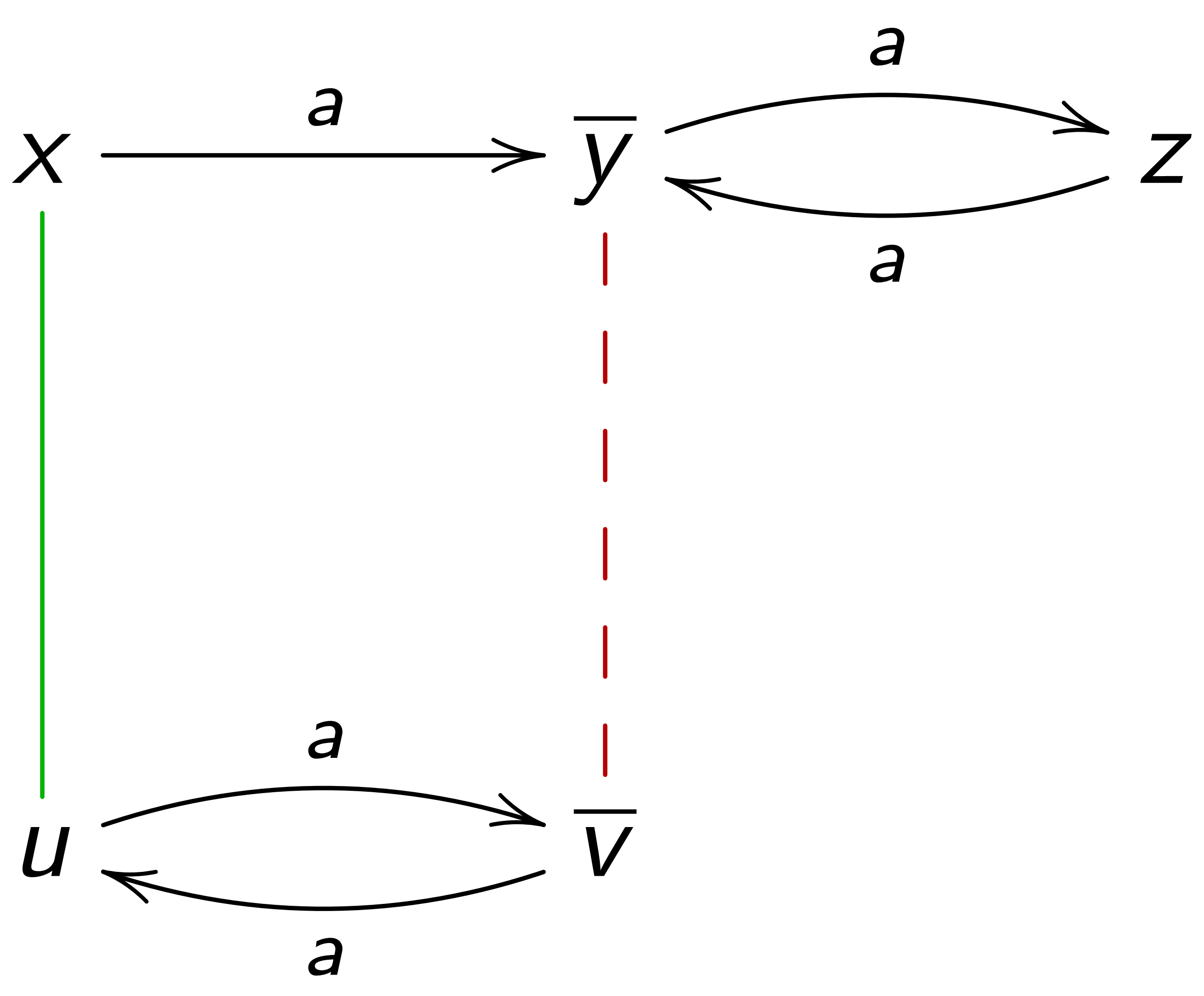

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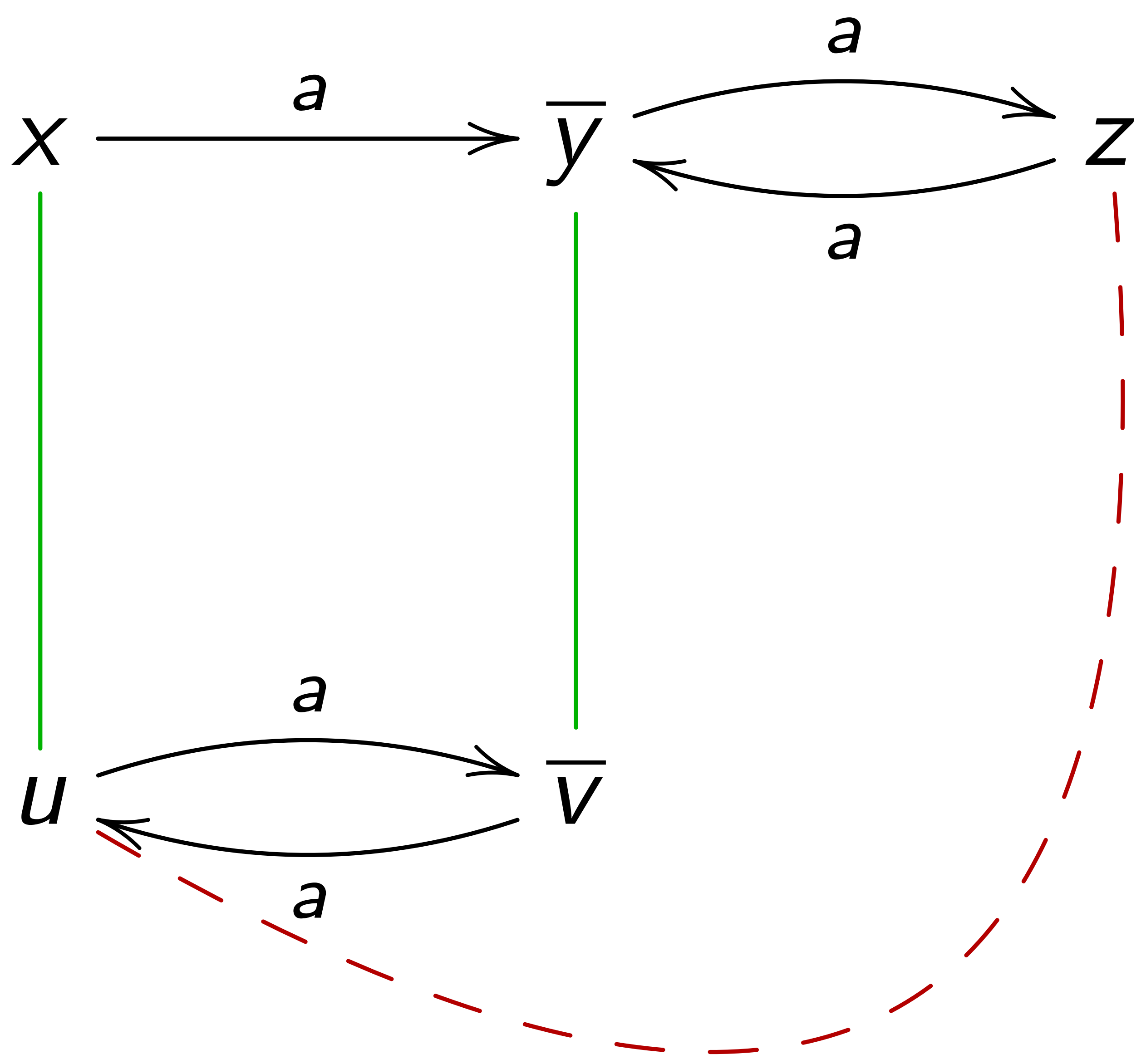

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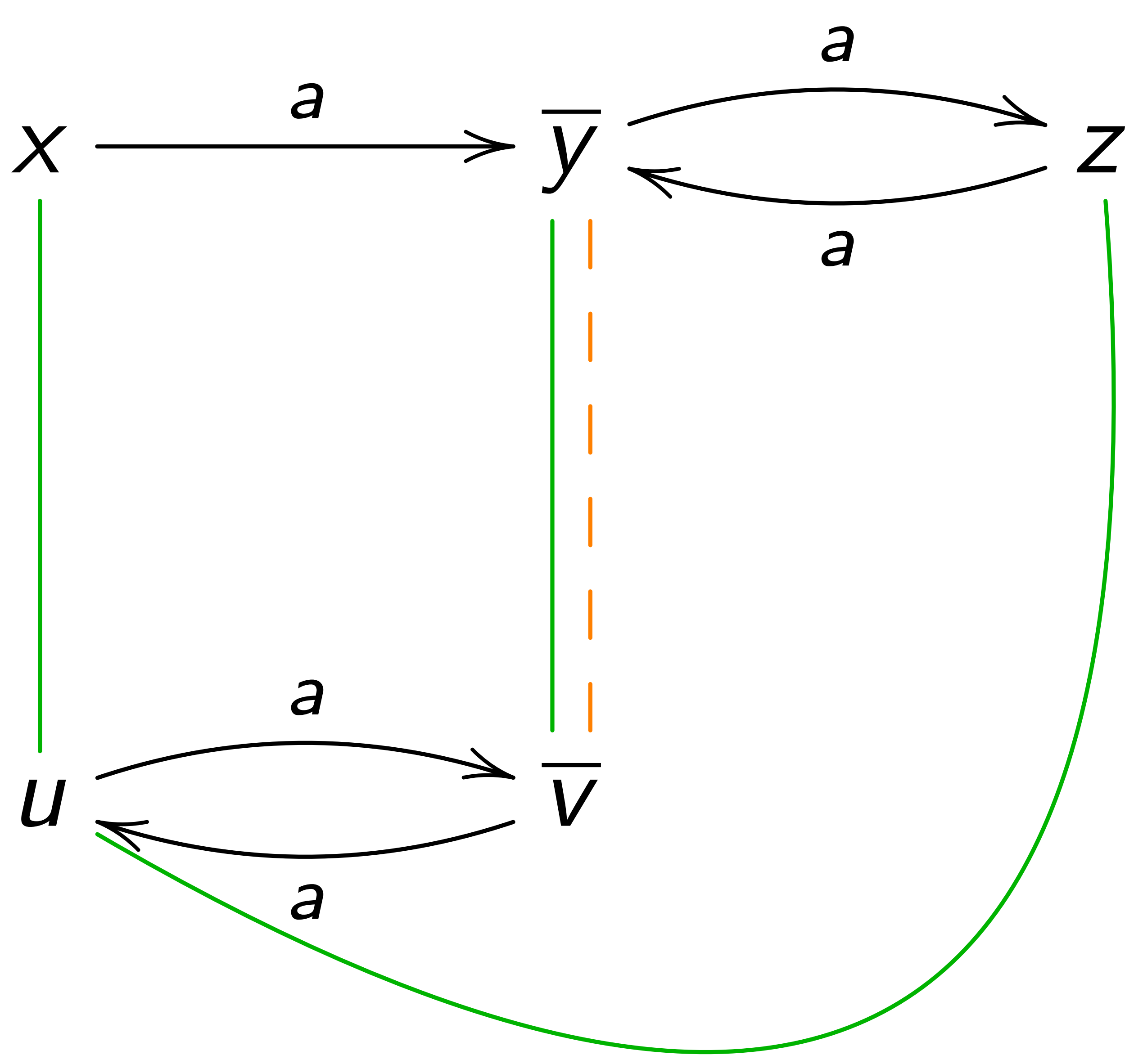

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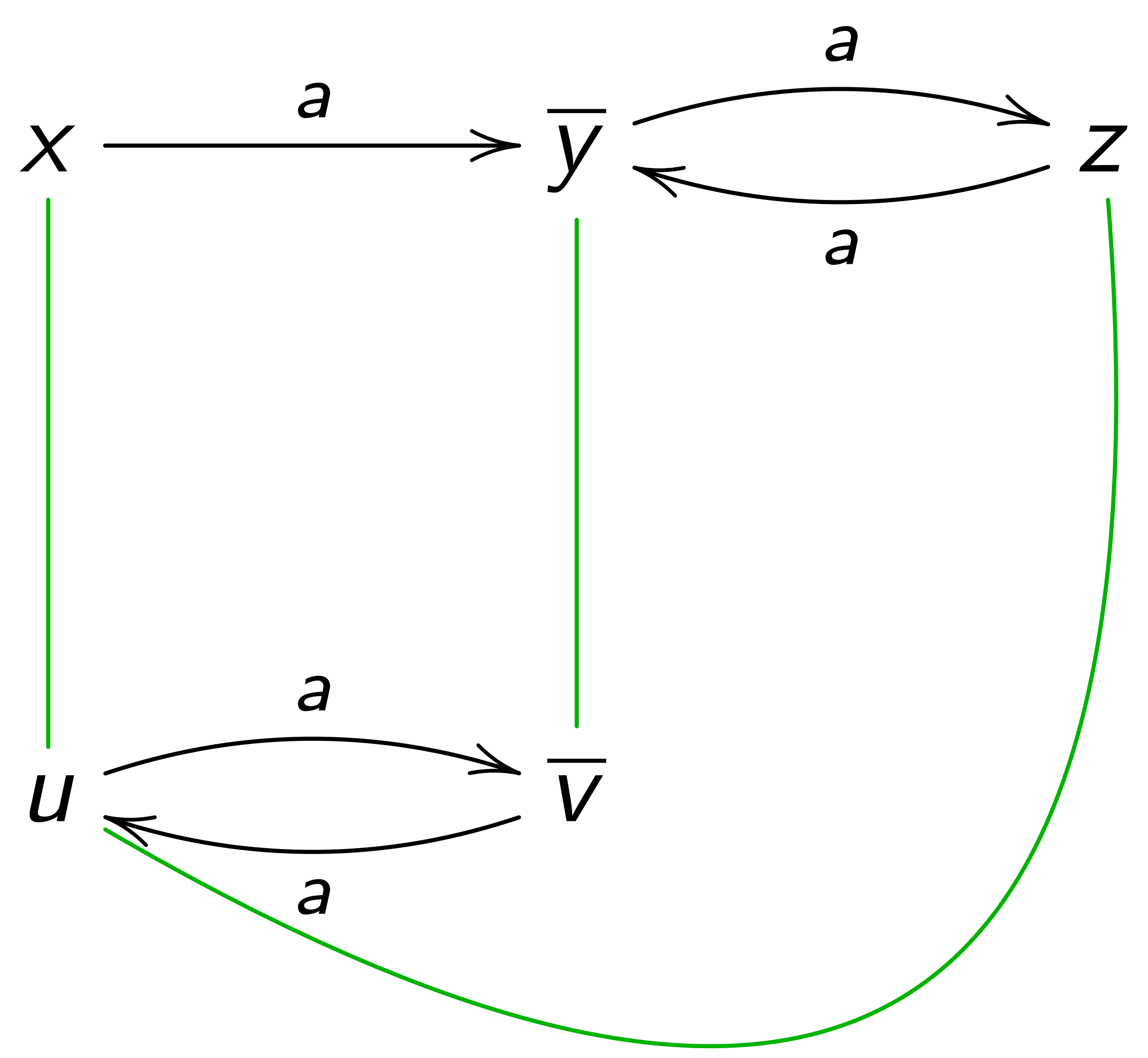
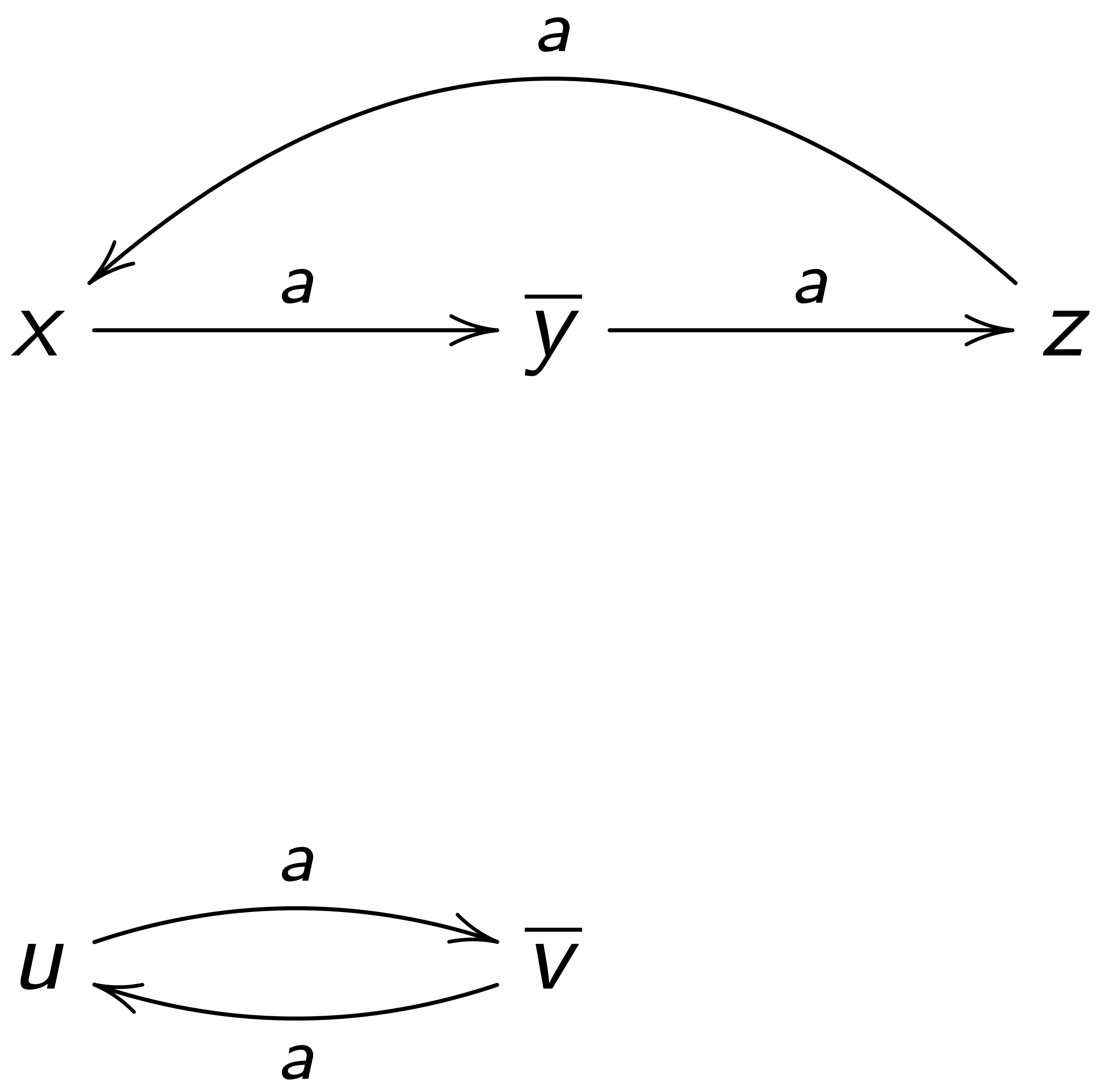


Deterministic finite automata

x and u are **not** equivalent



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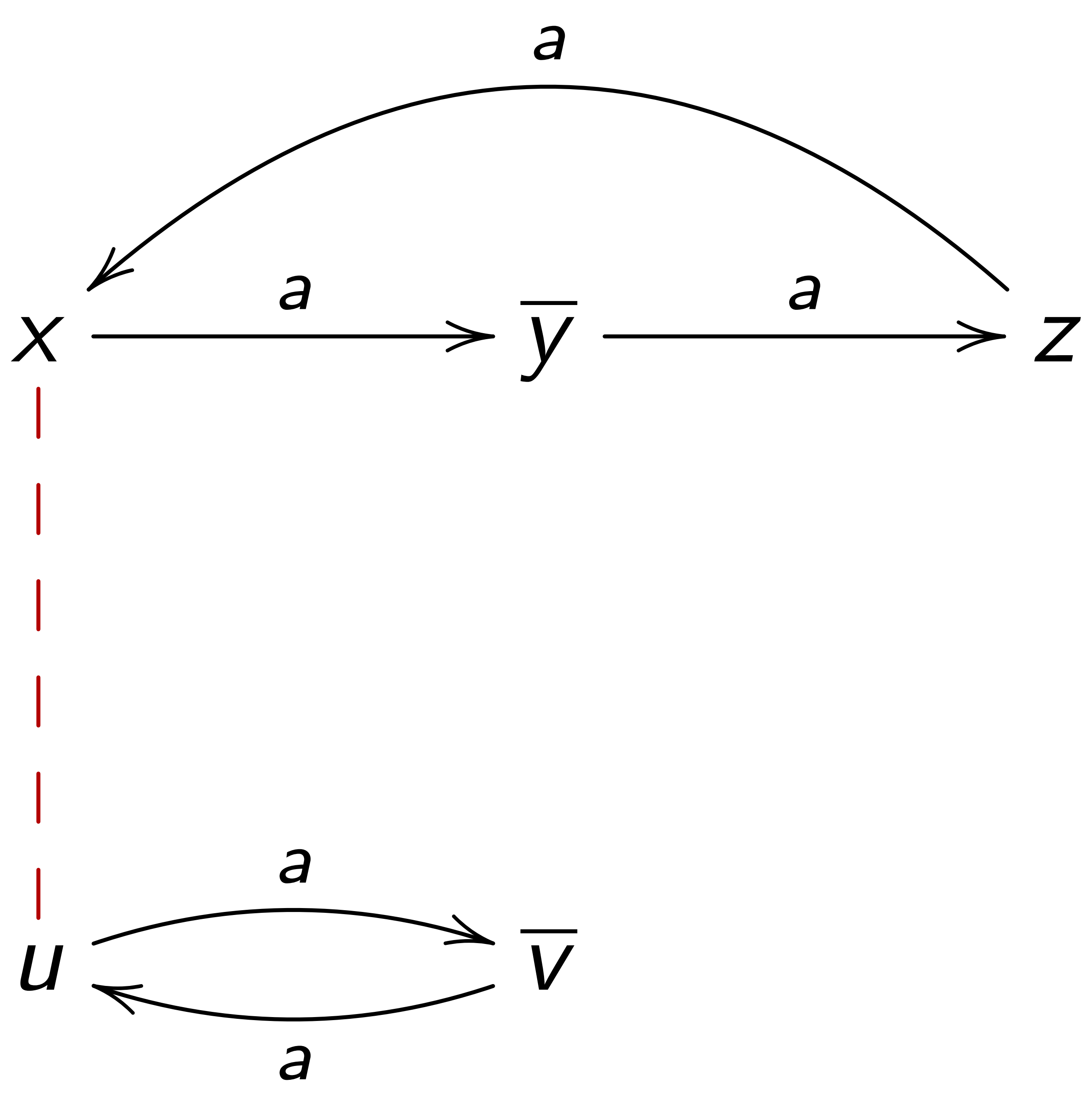

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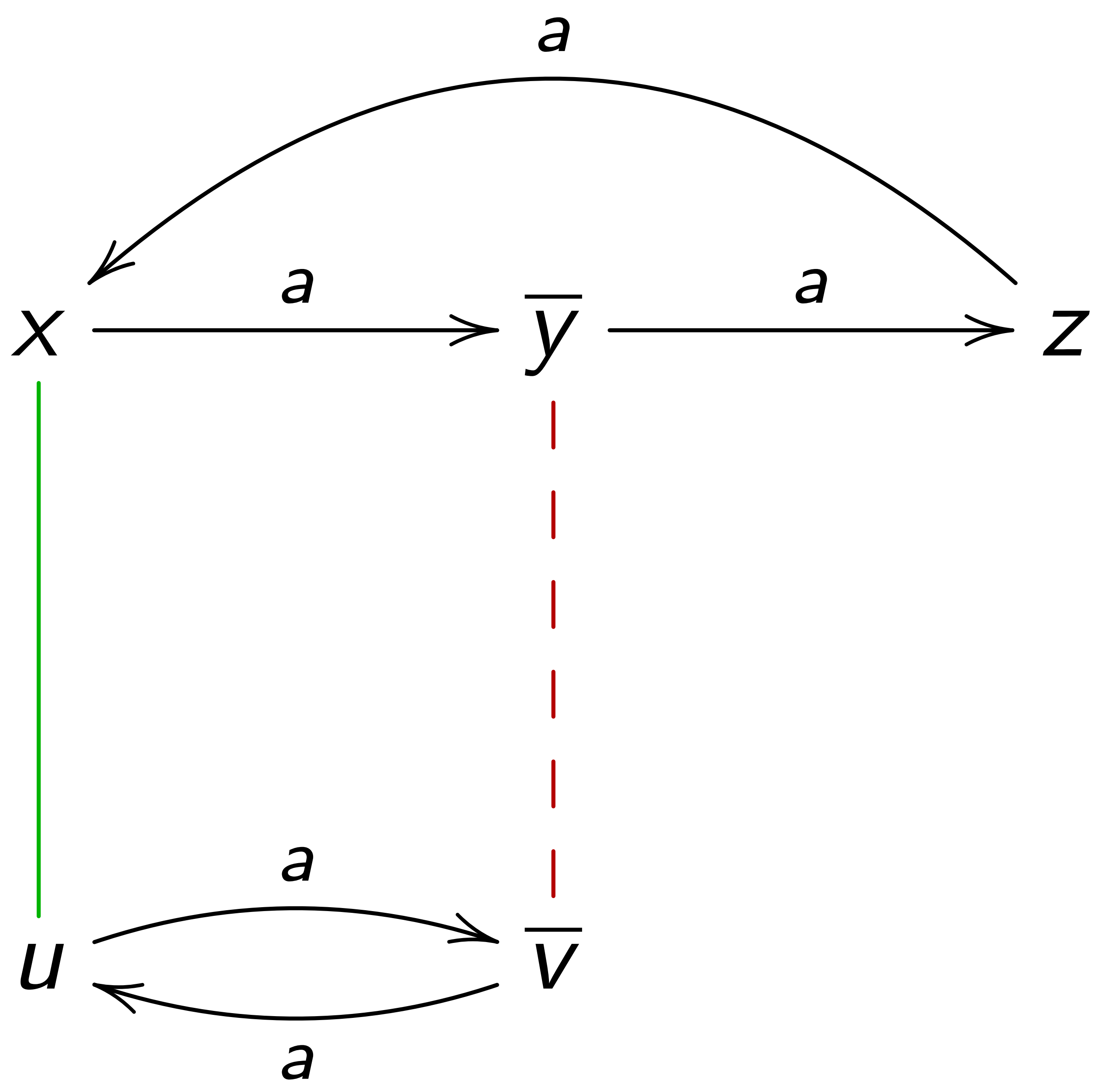

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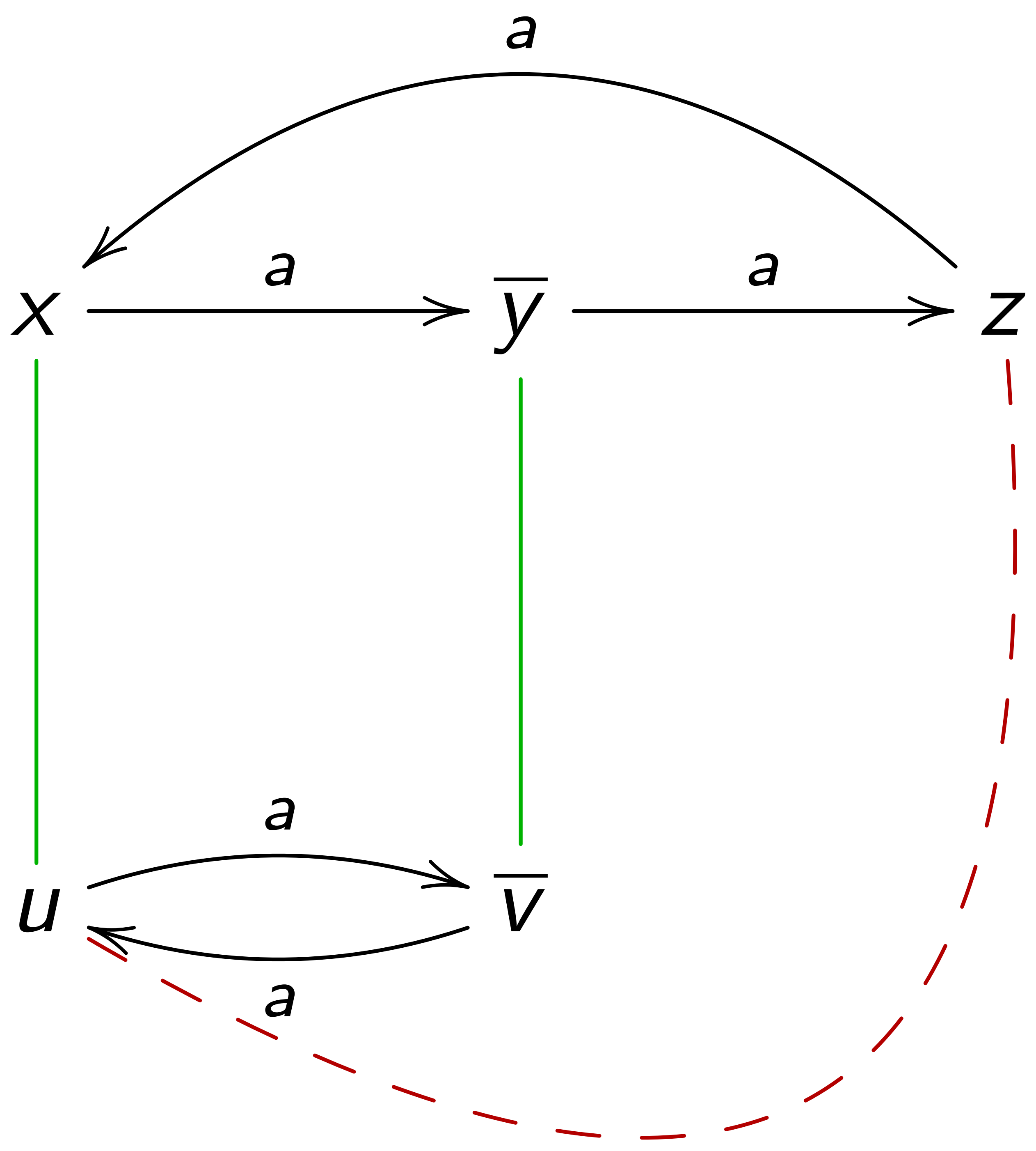

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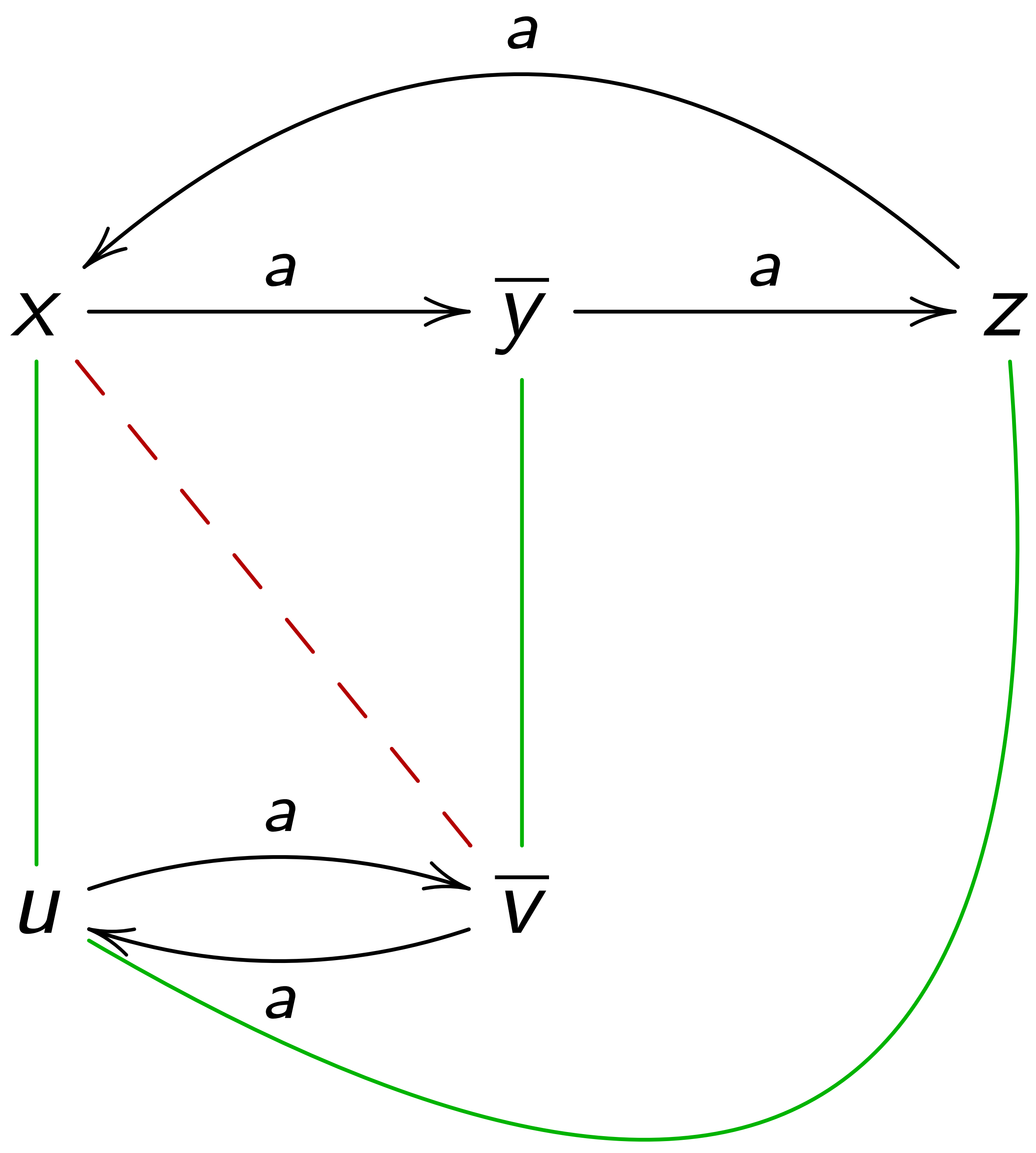

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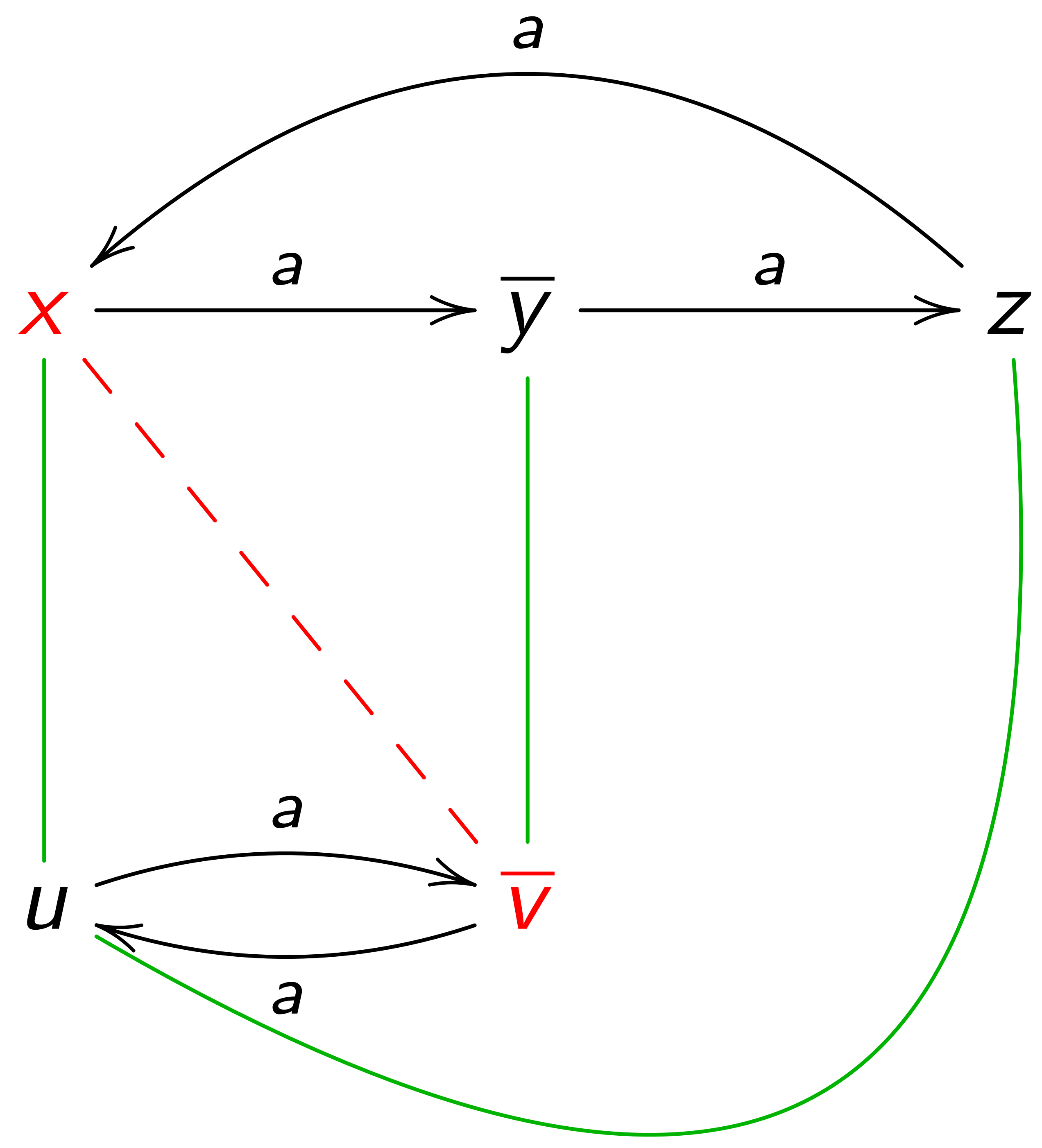
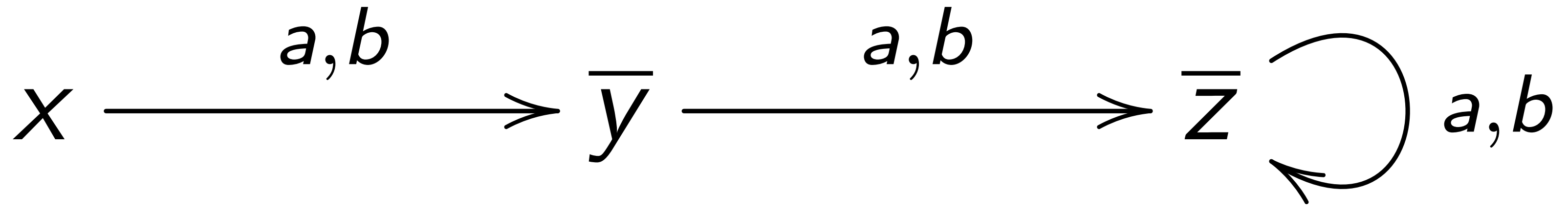
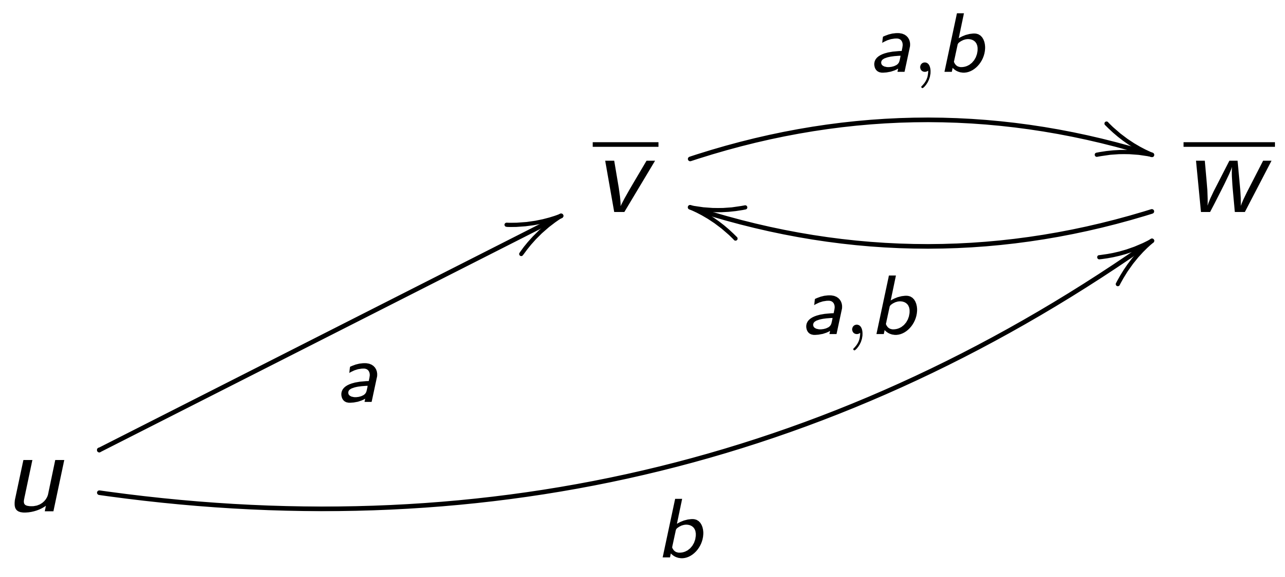


Deterministic finite automata

Last example, with two letters





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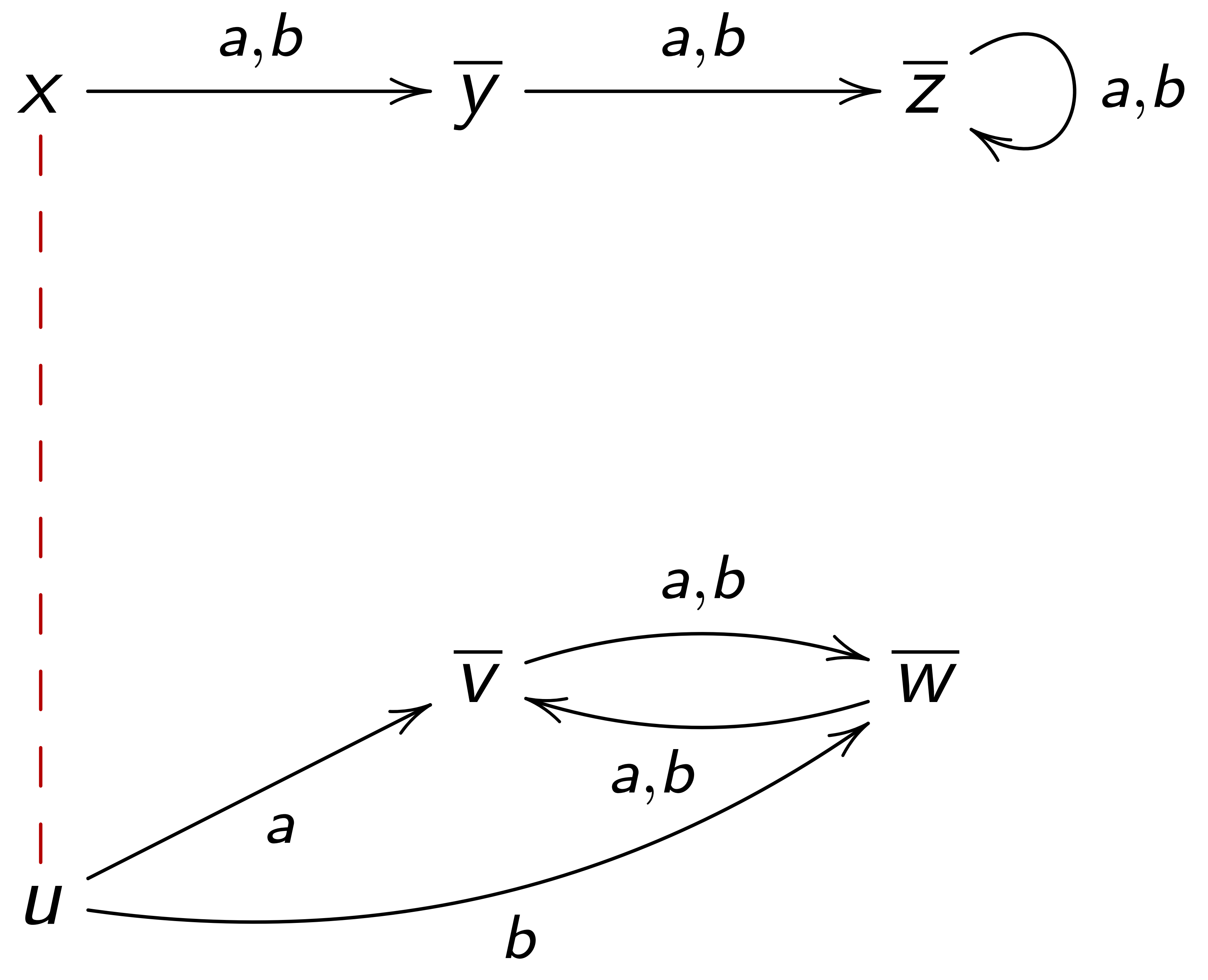

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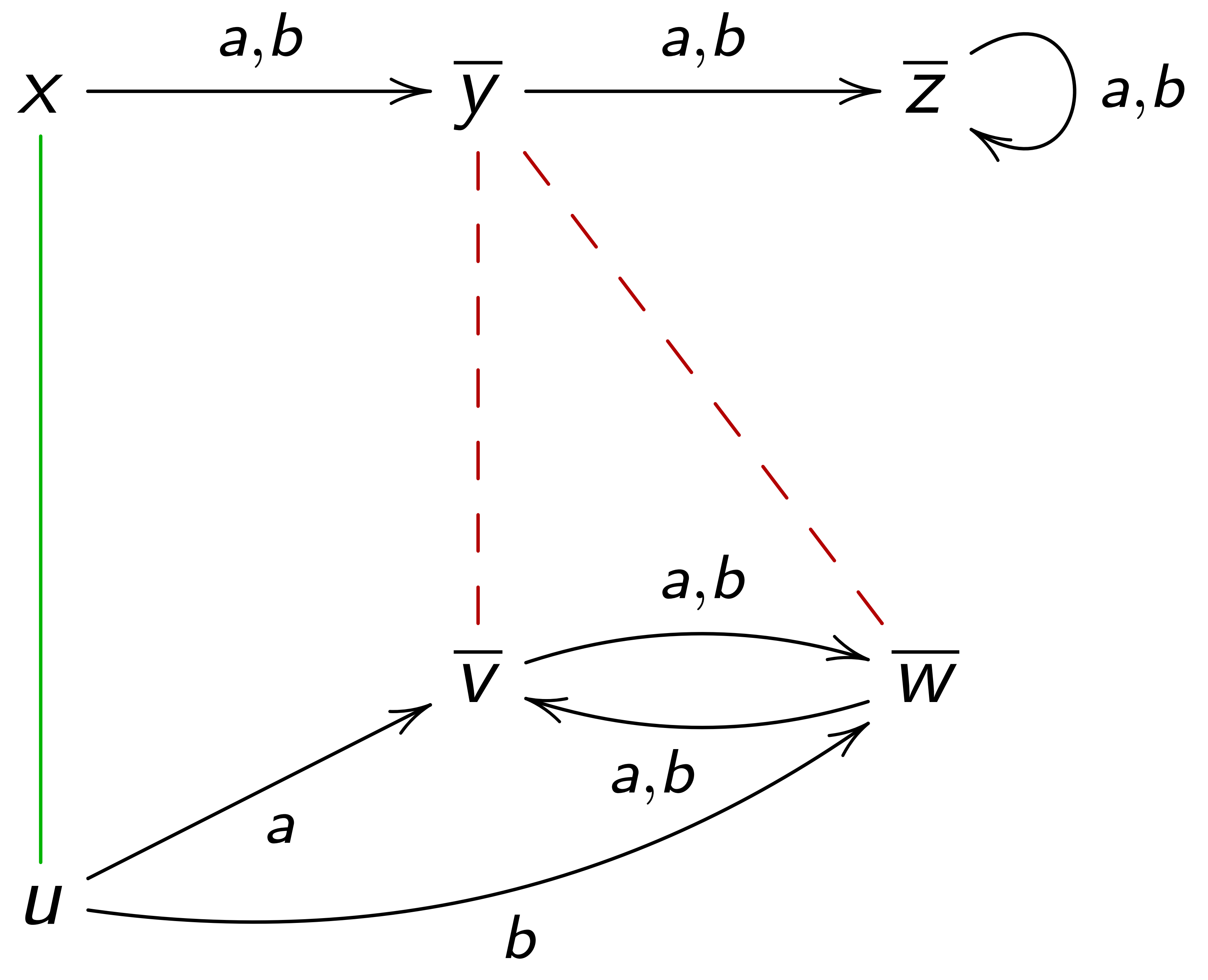

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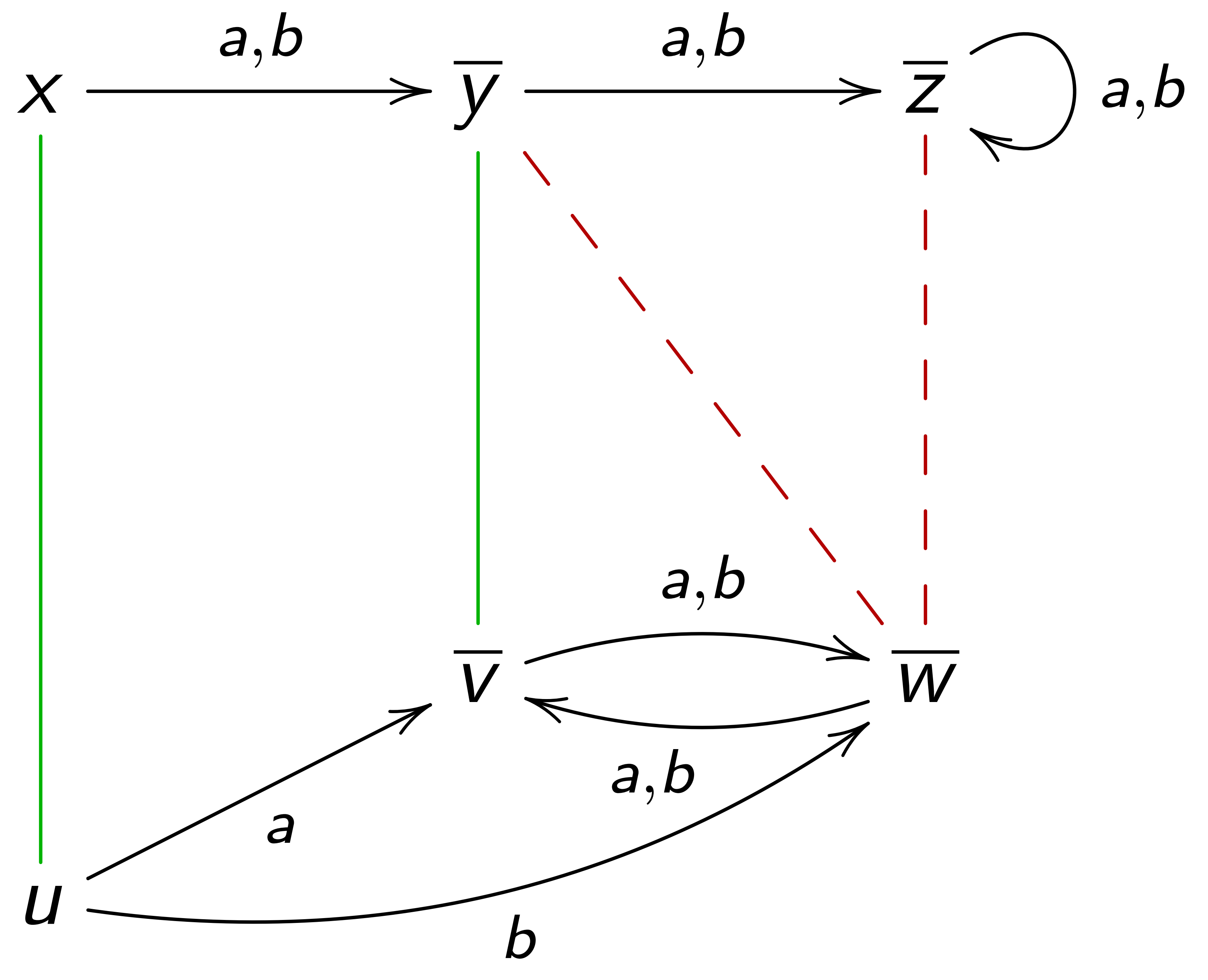

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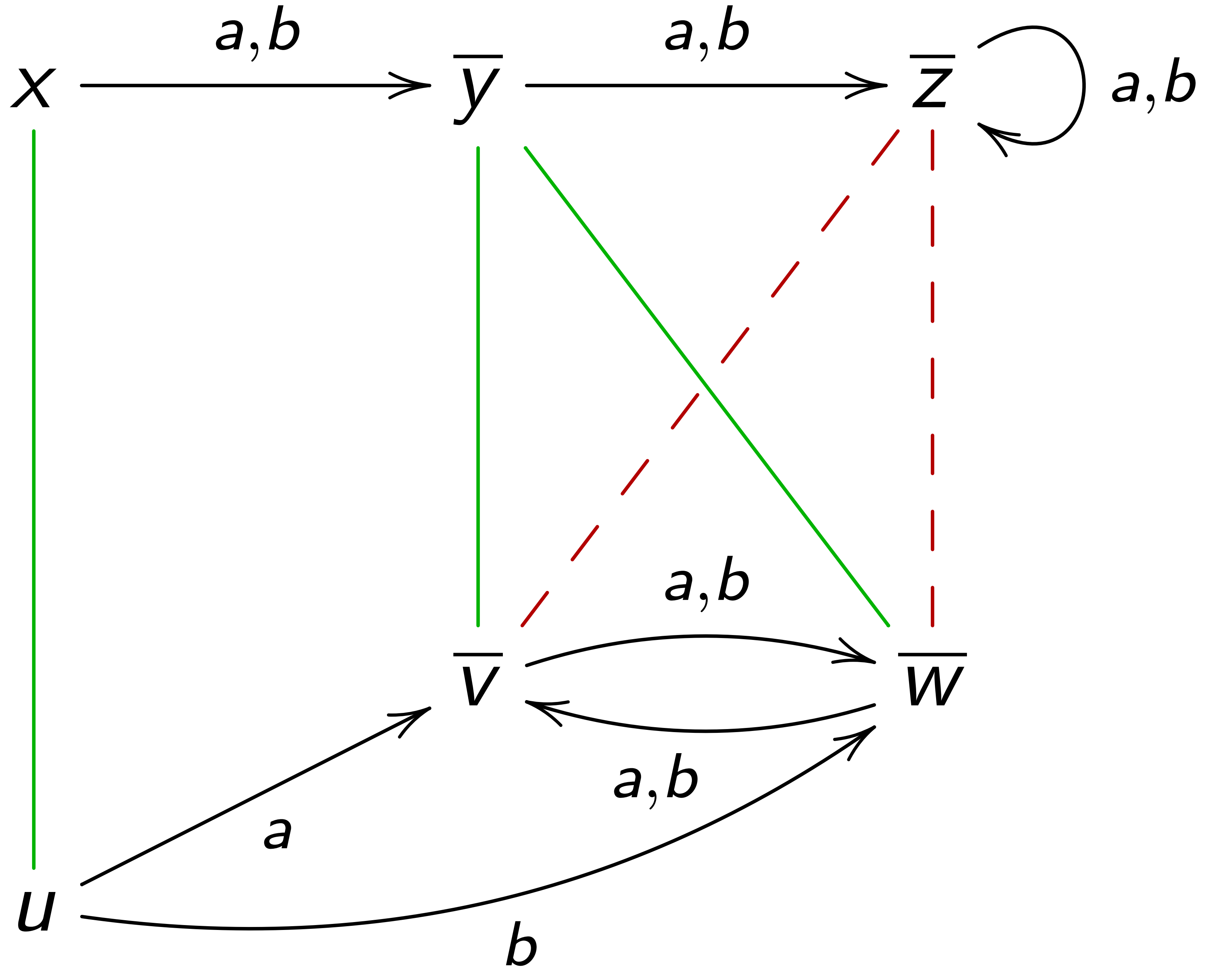

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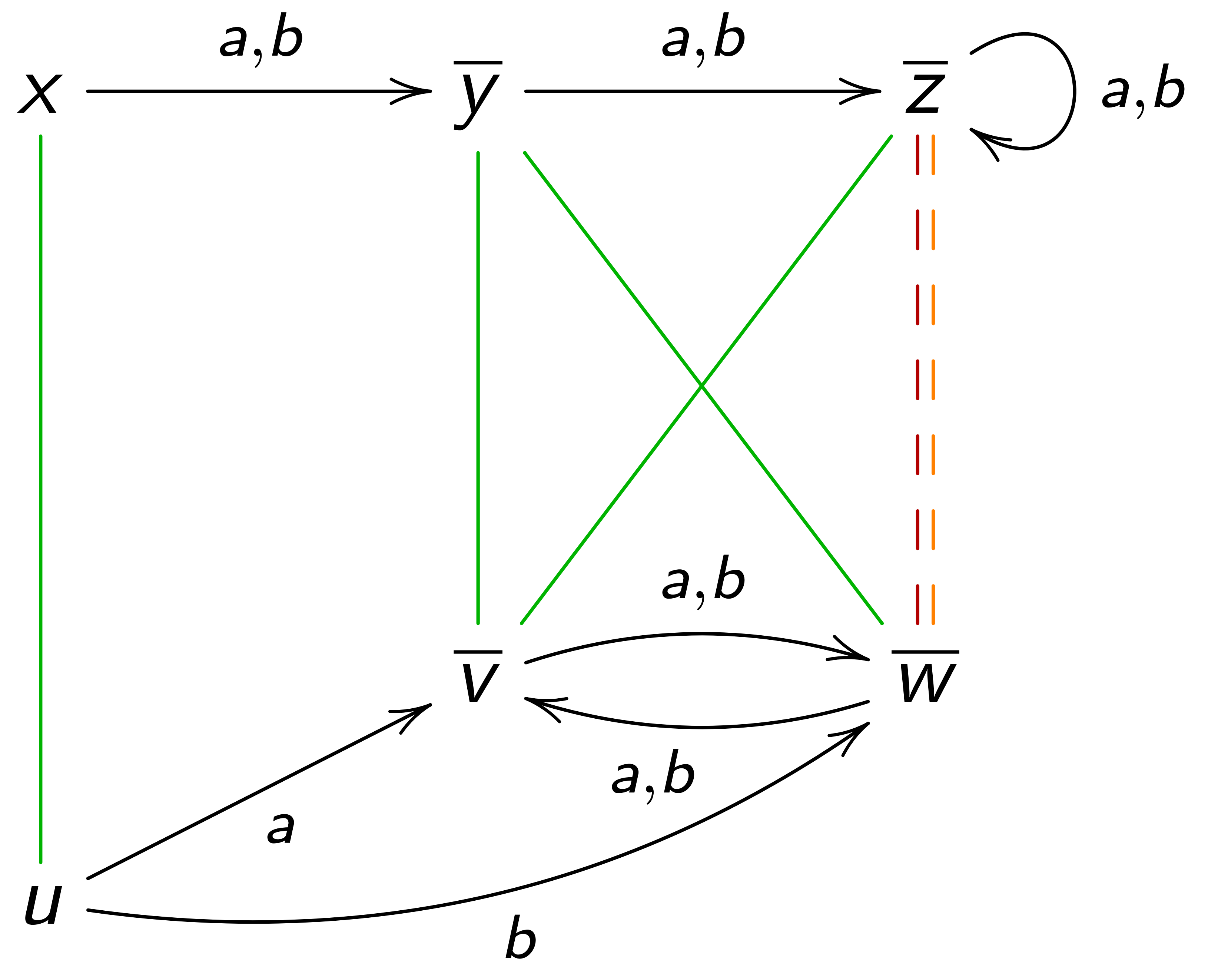

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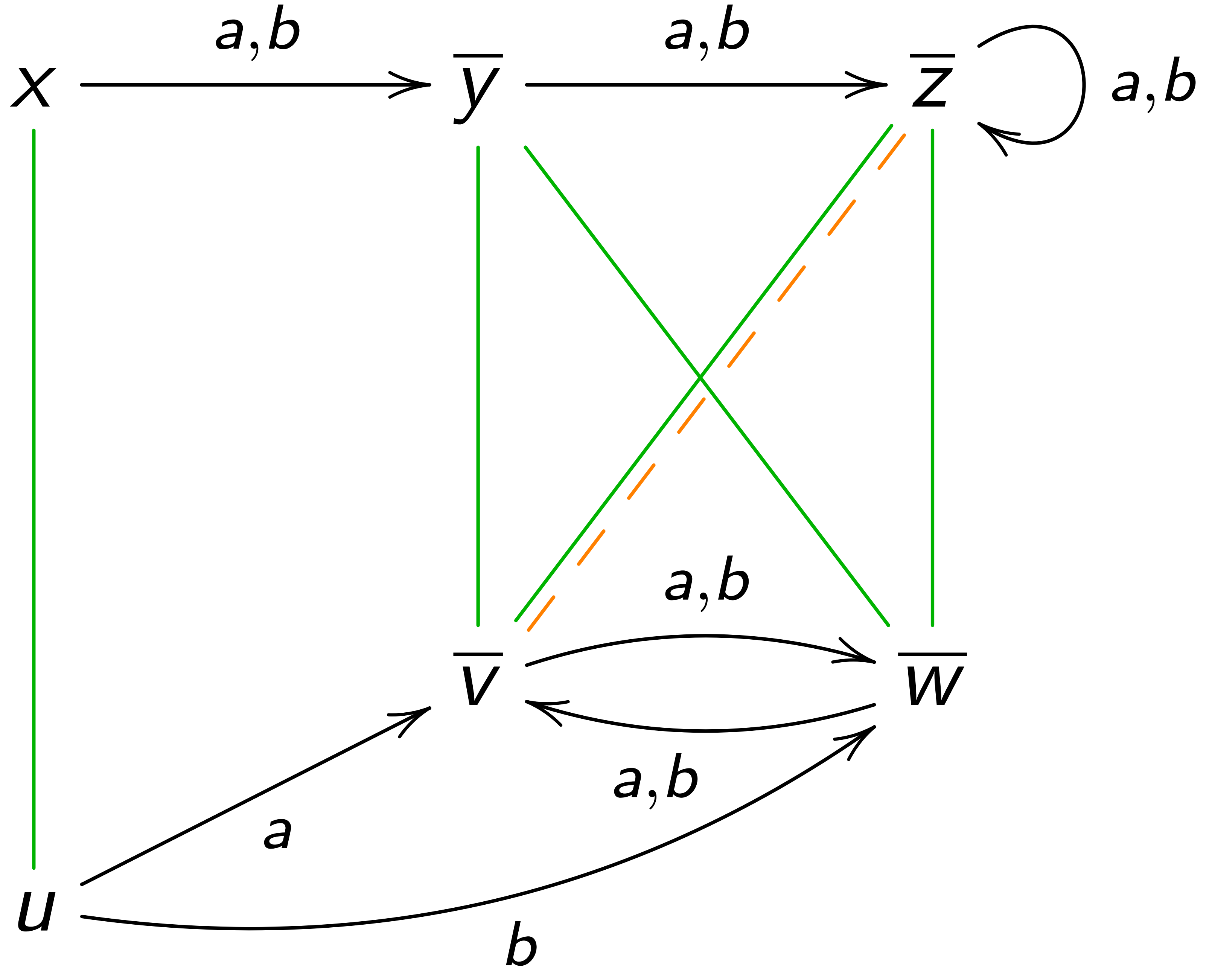

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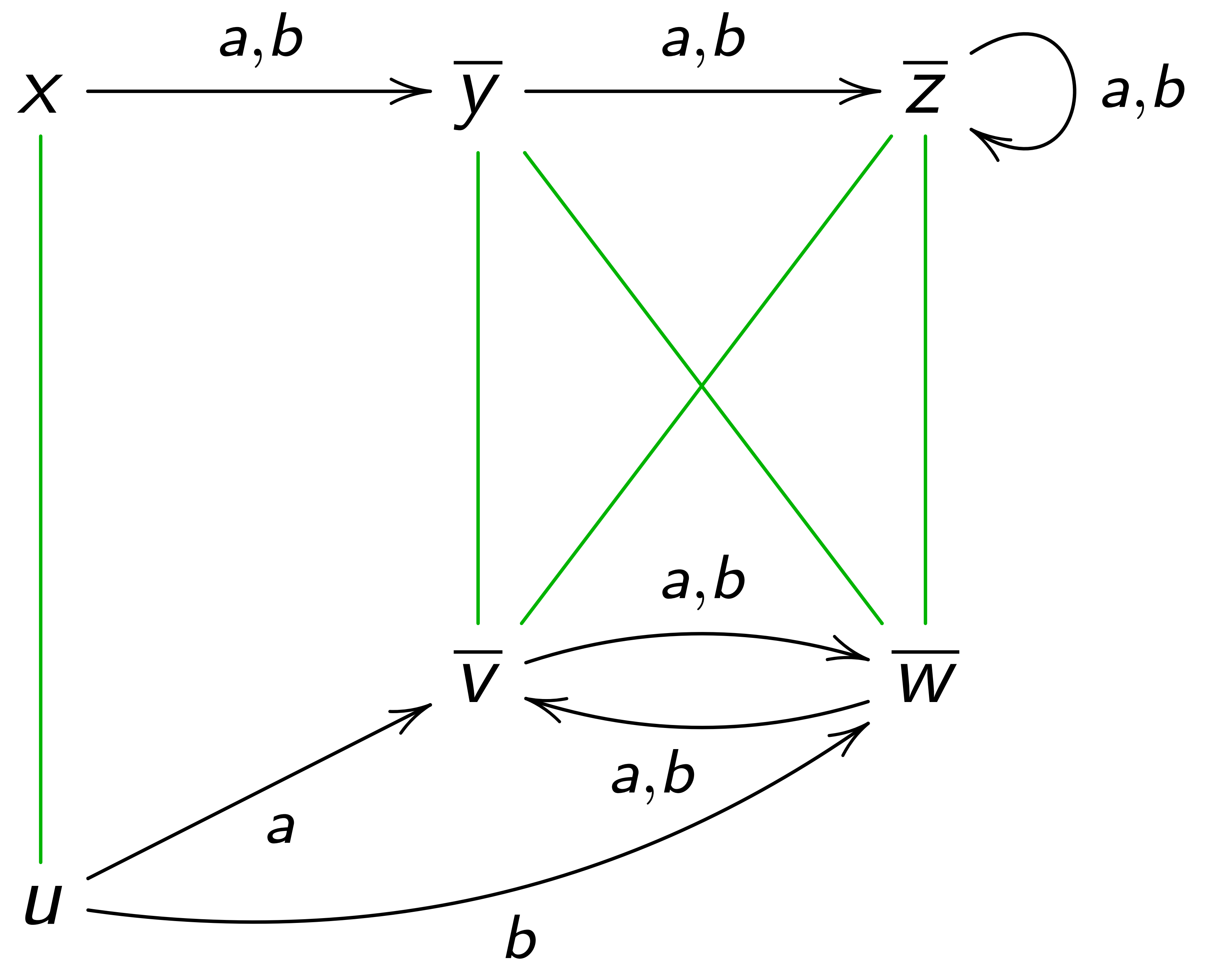

Correctness

- A relation R is a proof of equivalence (bisimulation) if $x \; R \; y$ entails
  - $o(x) = o(y)$;
  - for all a, $t_a(x) \; R \; t_a(y)$.

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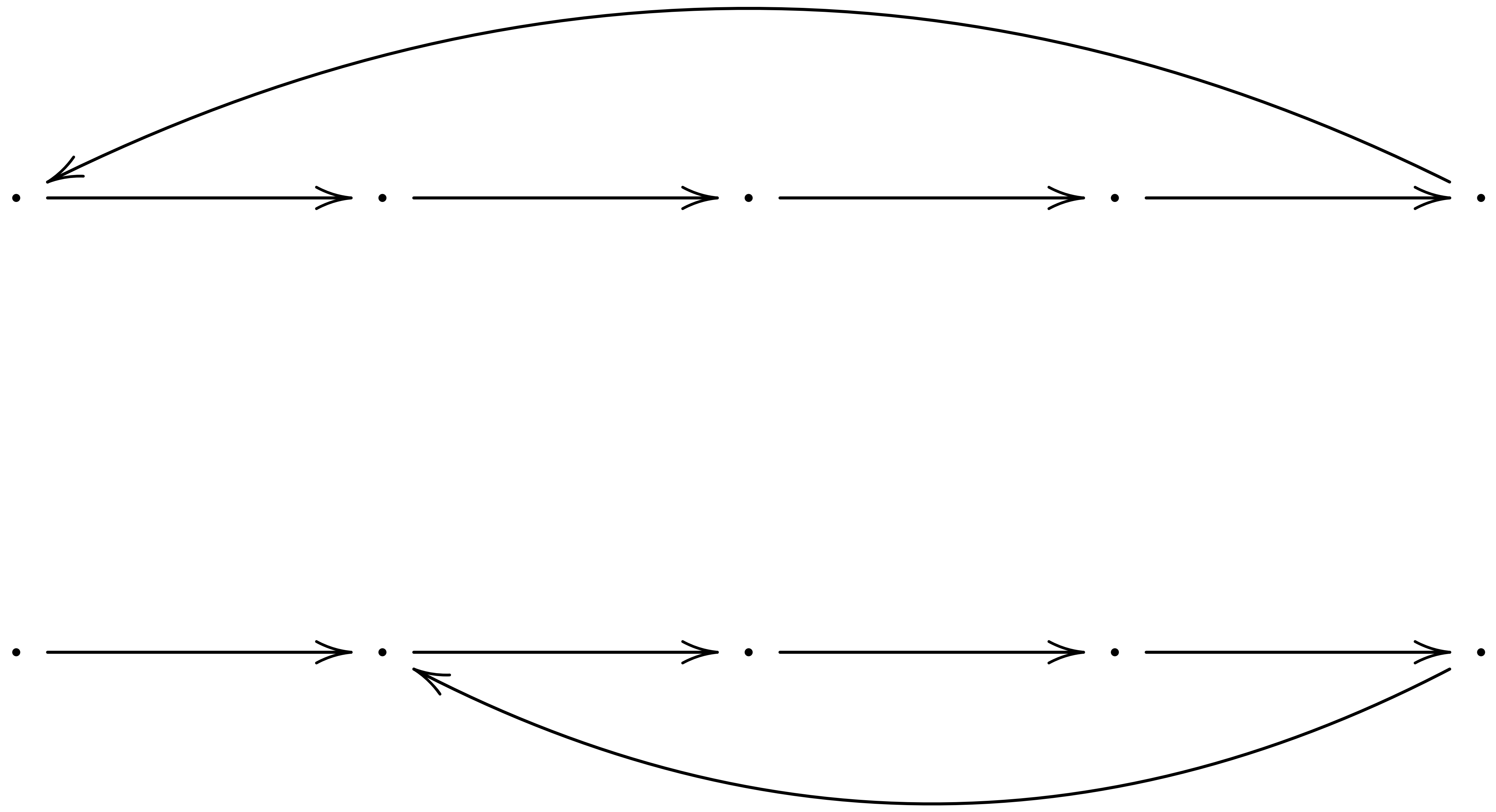
- *Theorem:* $L(x) = L(y)$ iff
  there exists a bisimulation R with $x \mathrel{R} y$

The previous algorithm attempts to construct a bisimulation

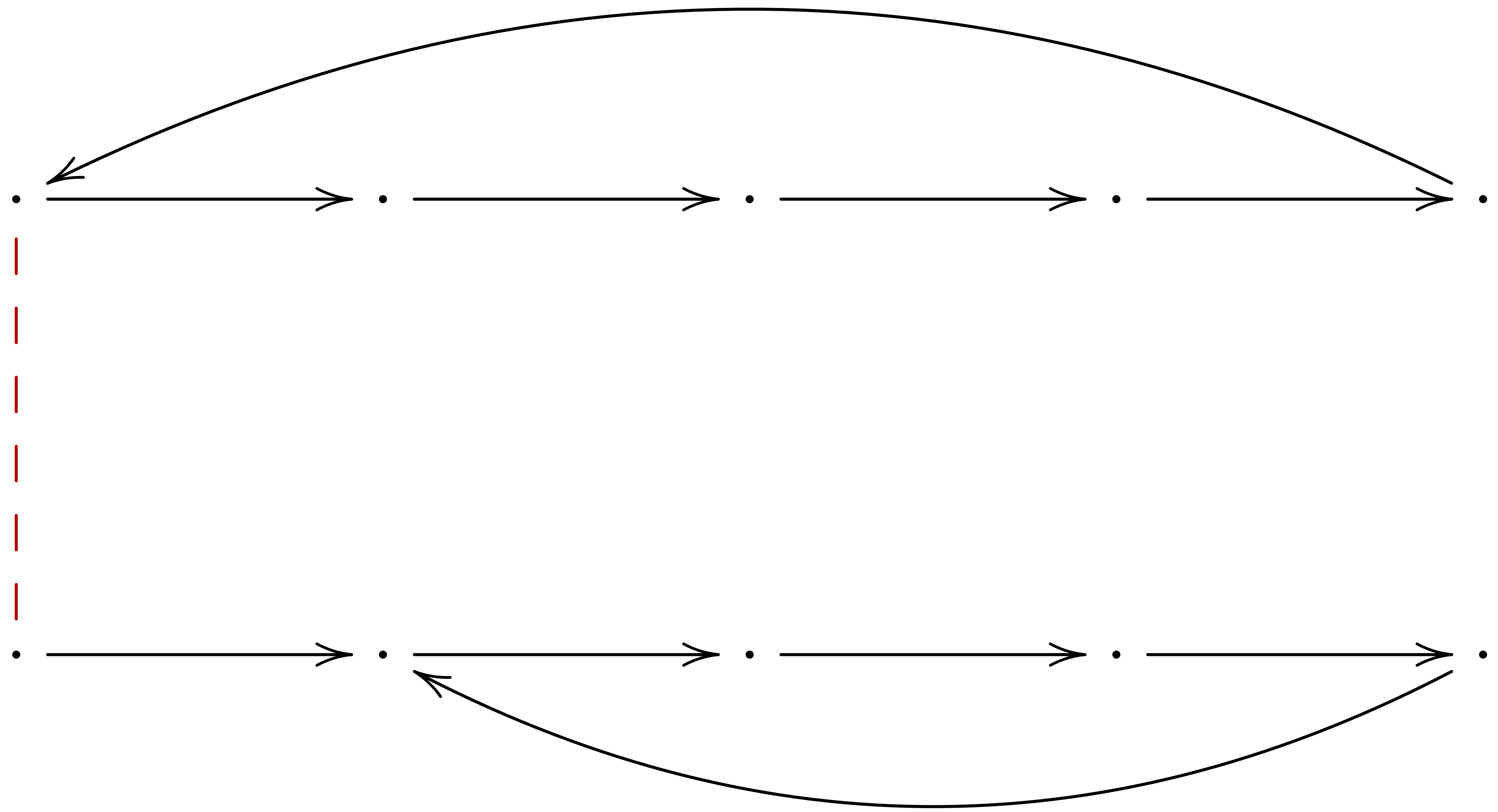
Complexity

The previous algorithm is quadratic



Complexity

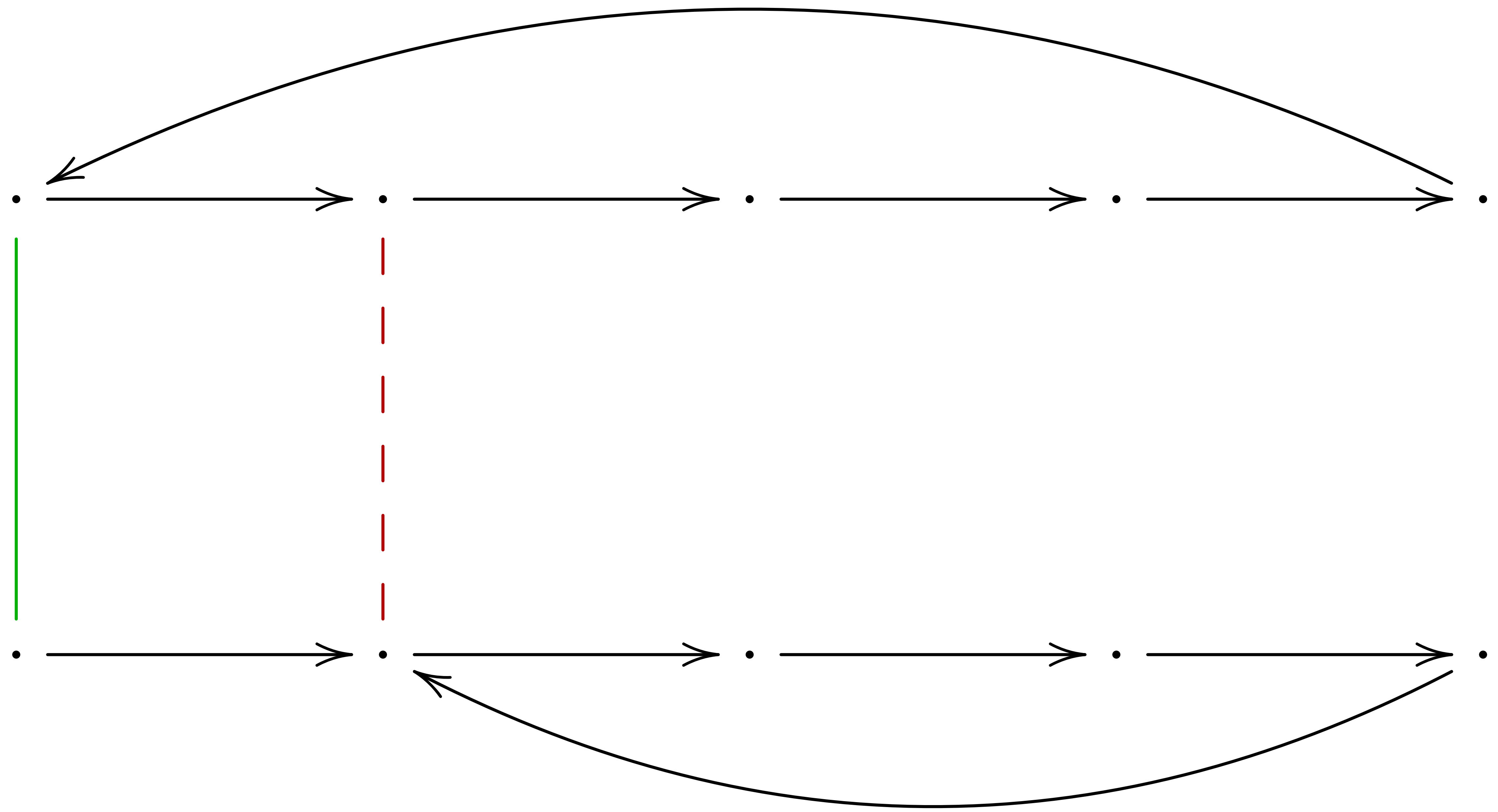
The previous algorithm is quadratic



0 pairs

Complexity

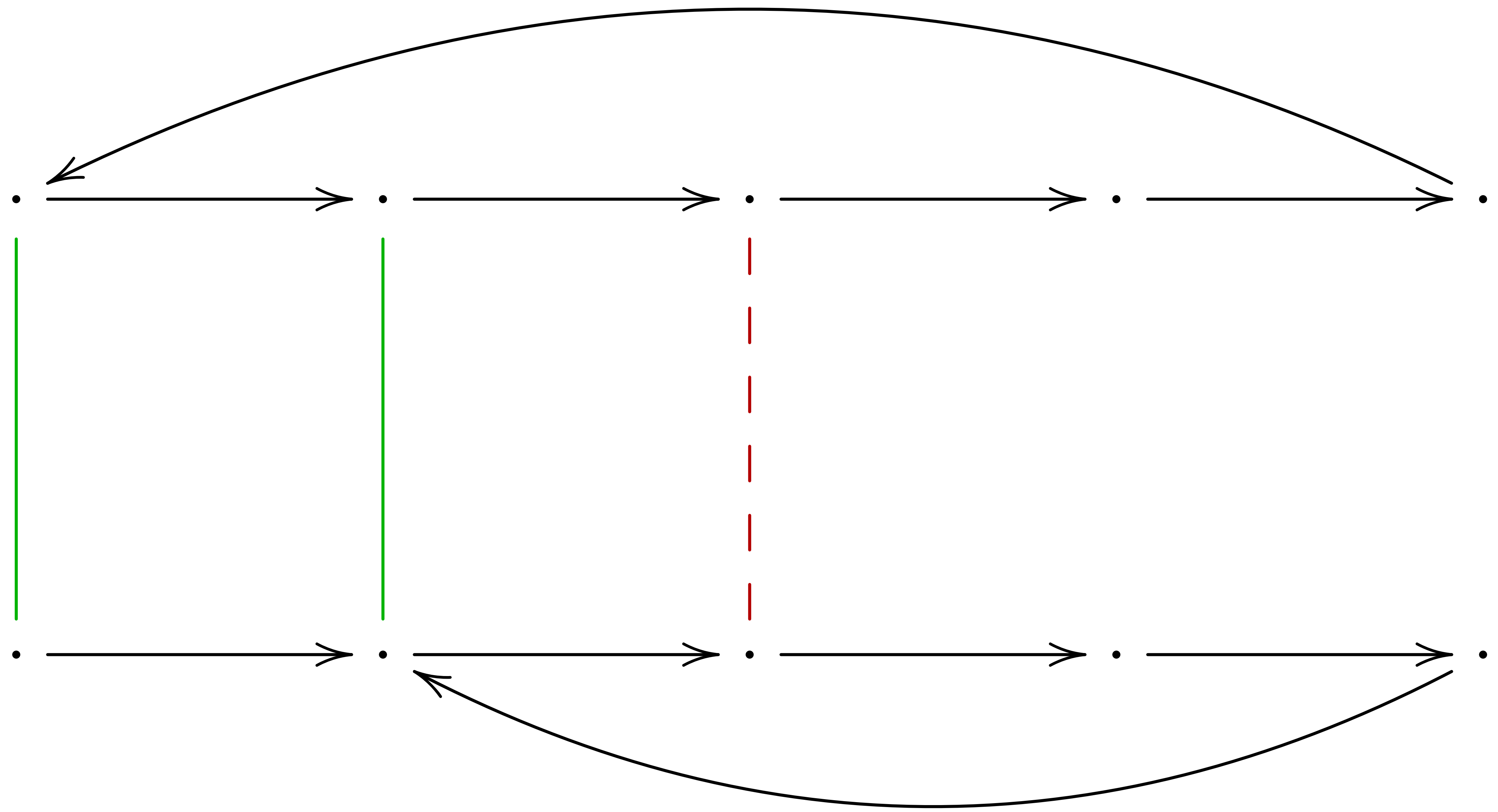
The previous algorithm is quadratic



1 pairs

Complexity

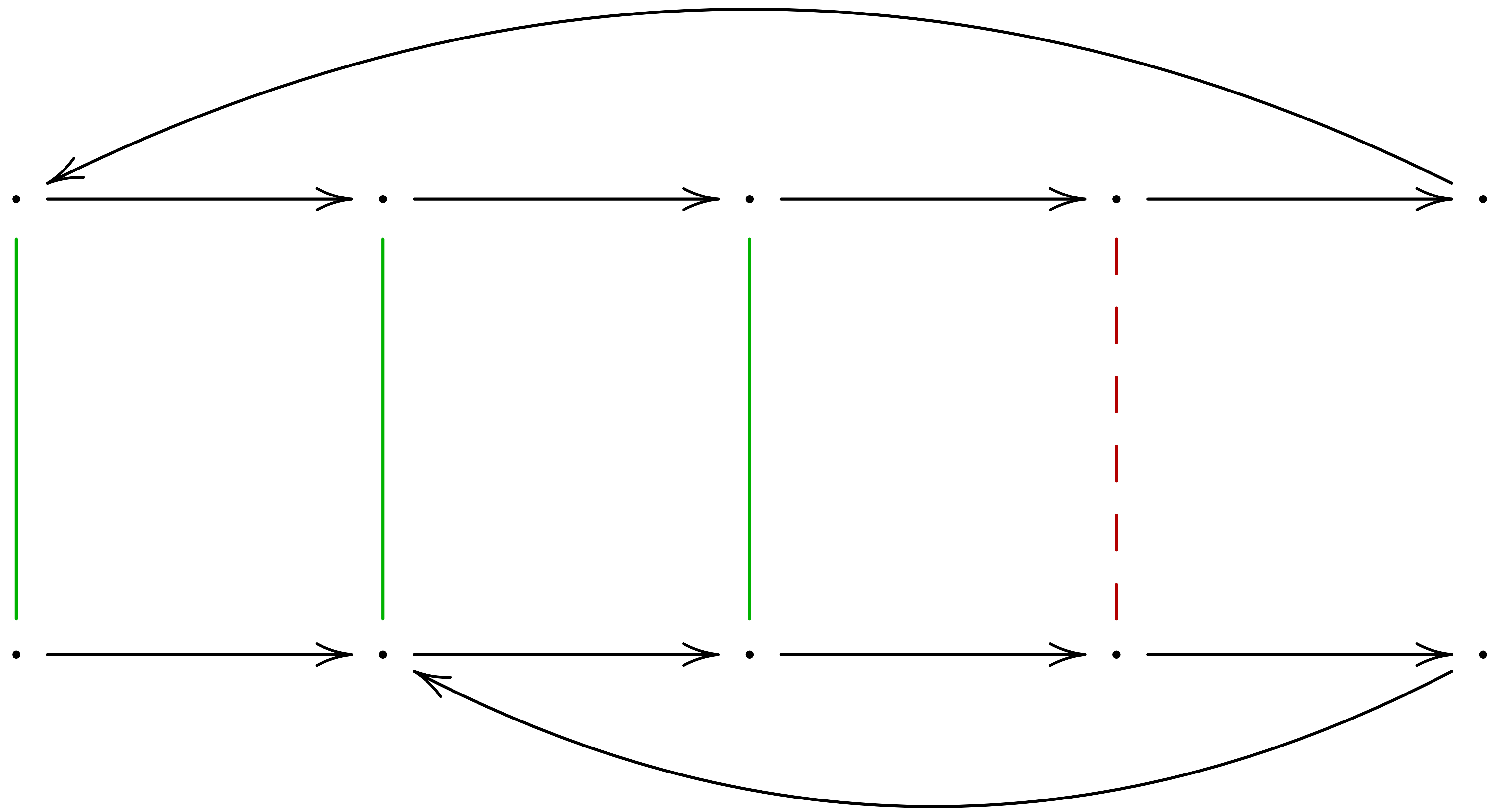
The previous algorithm is quadratic



2 pairs

Complexity

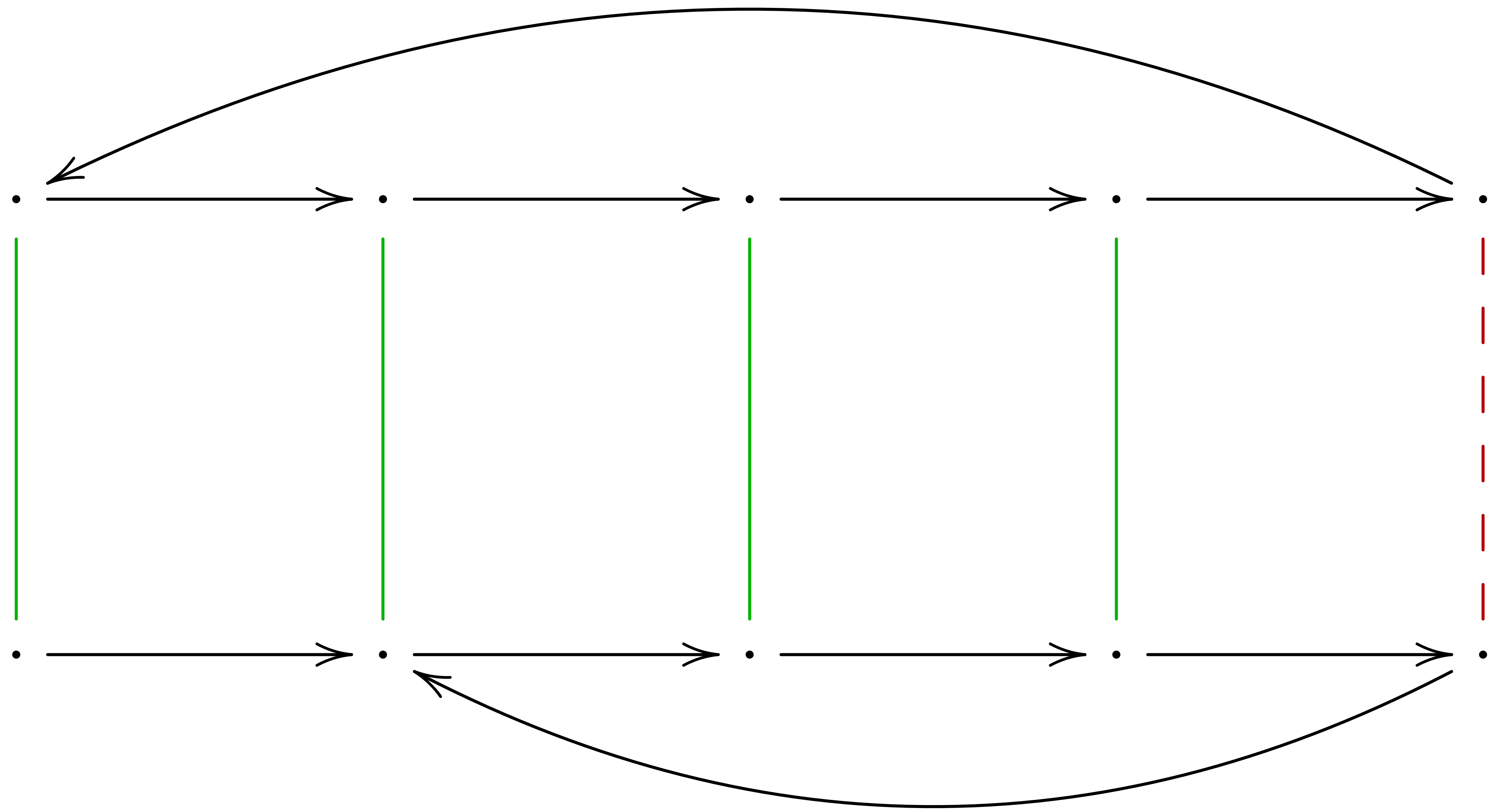
The previous algorithm is quadratic



3 pairs

Complexity

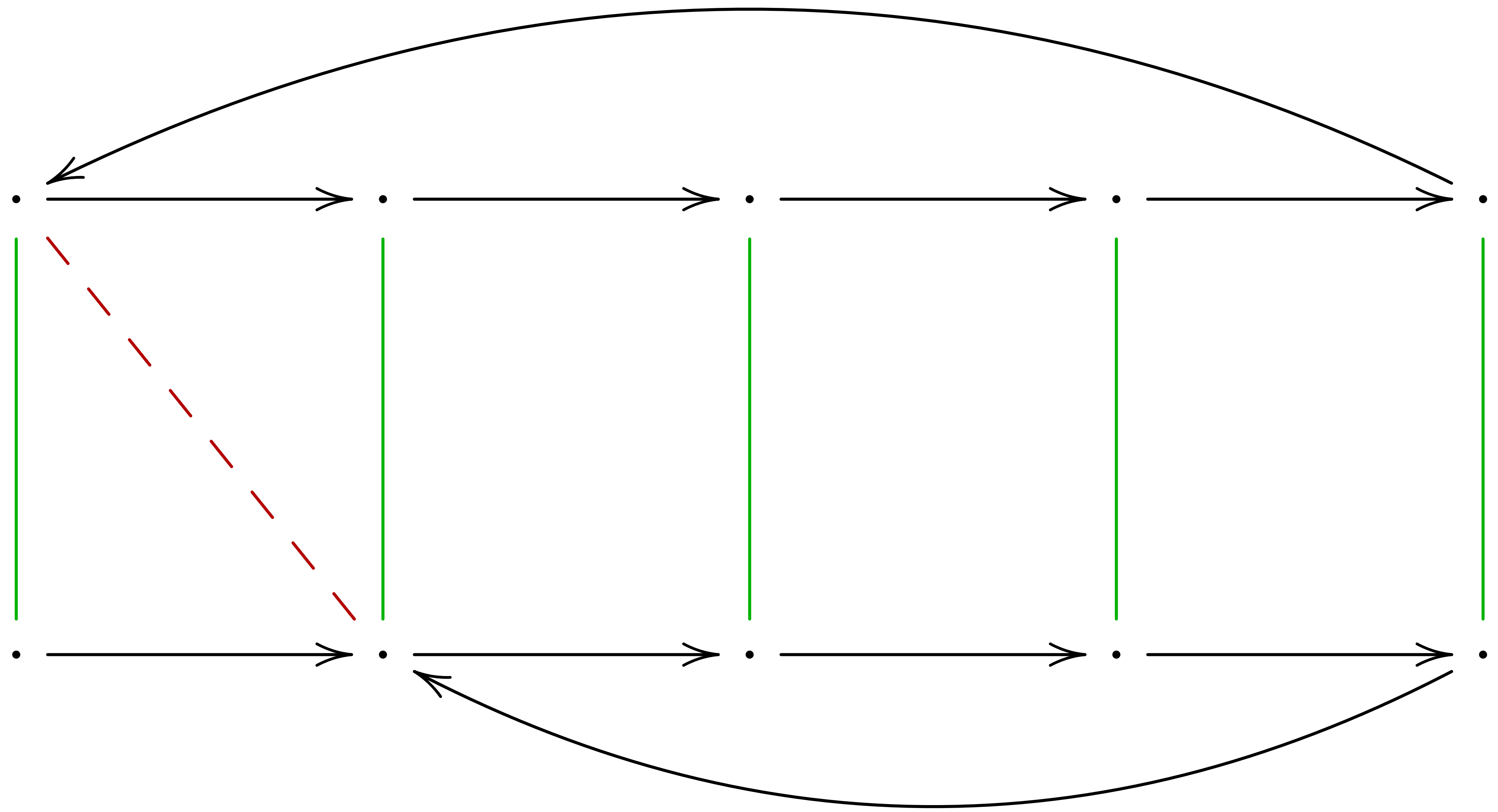
The previous algorithm is quadratic



4 pairs

Complexity

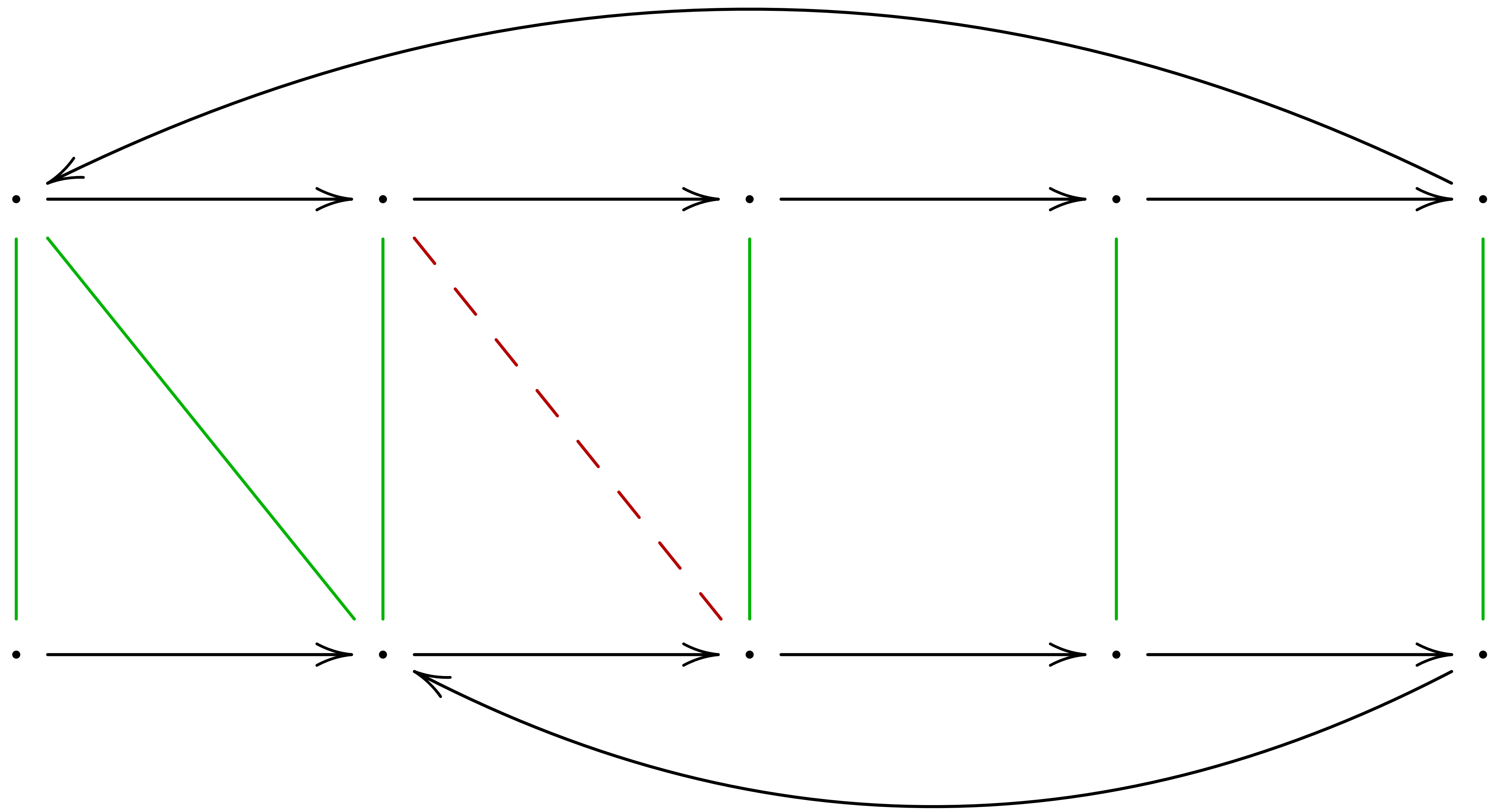
The previous algorithm is quadratic



5 pairs

Complexity

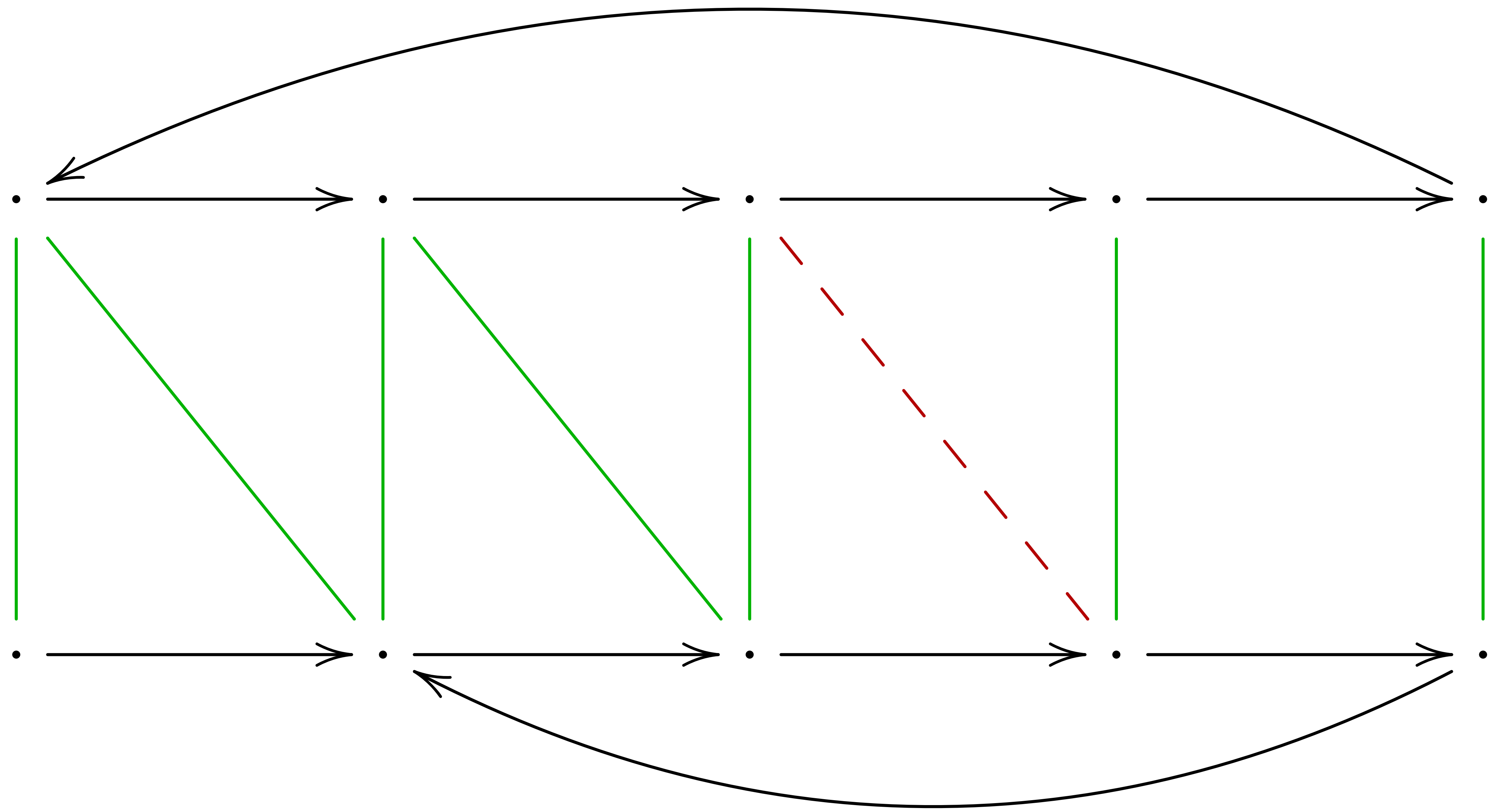
The previous algorithm is quadratic



6 pairs

Complexity

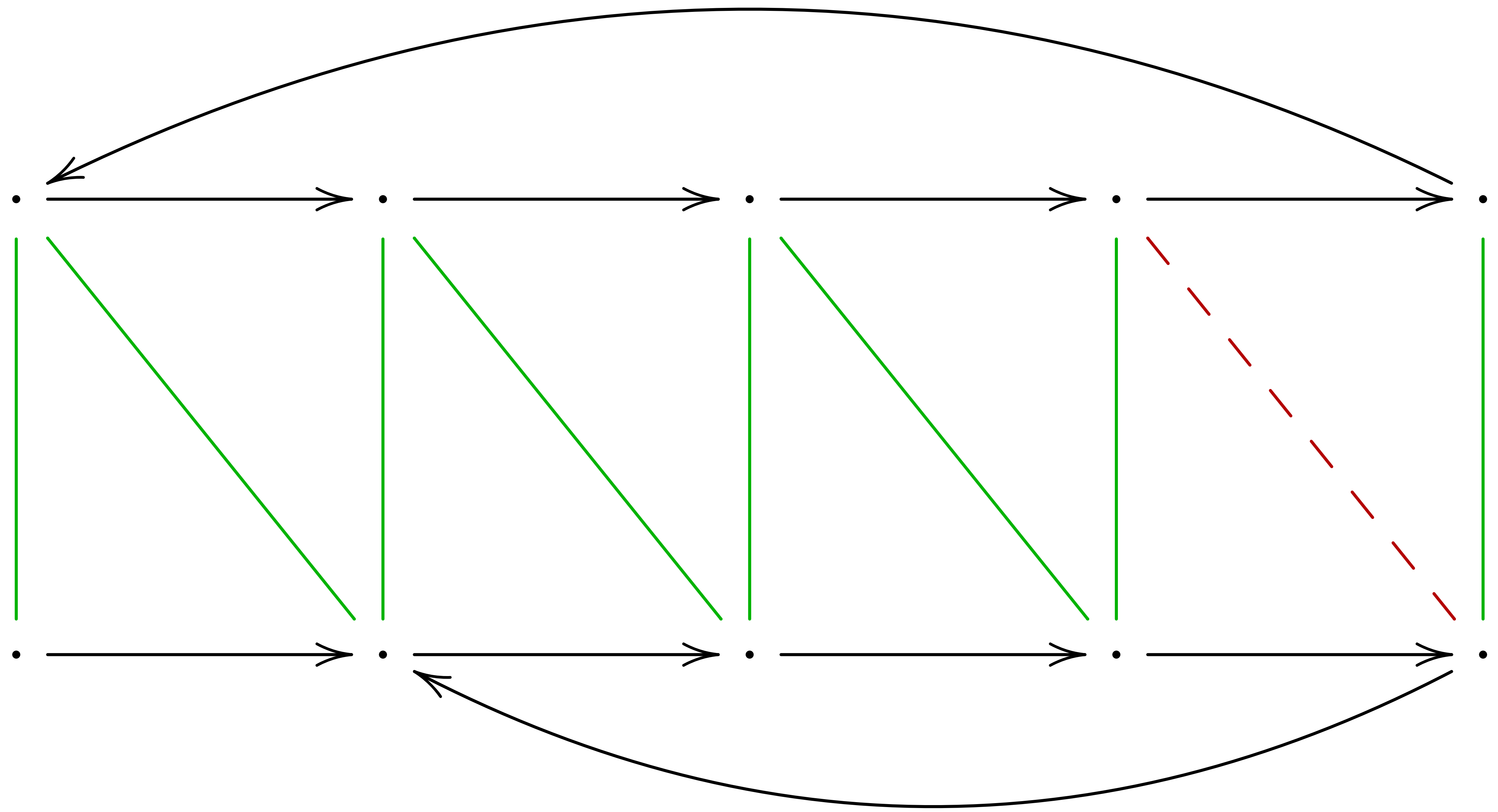
The previous algorithm is quadratic



7 pairs

Complexity

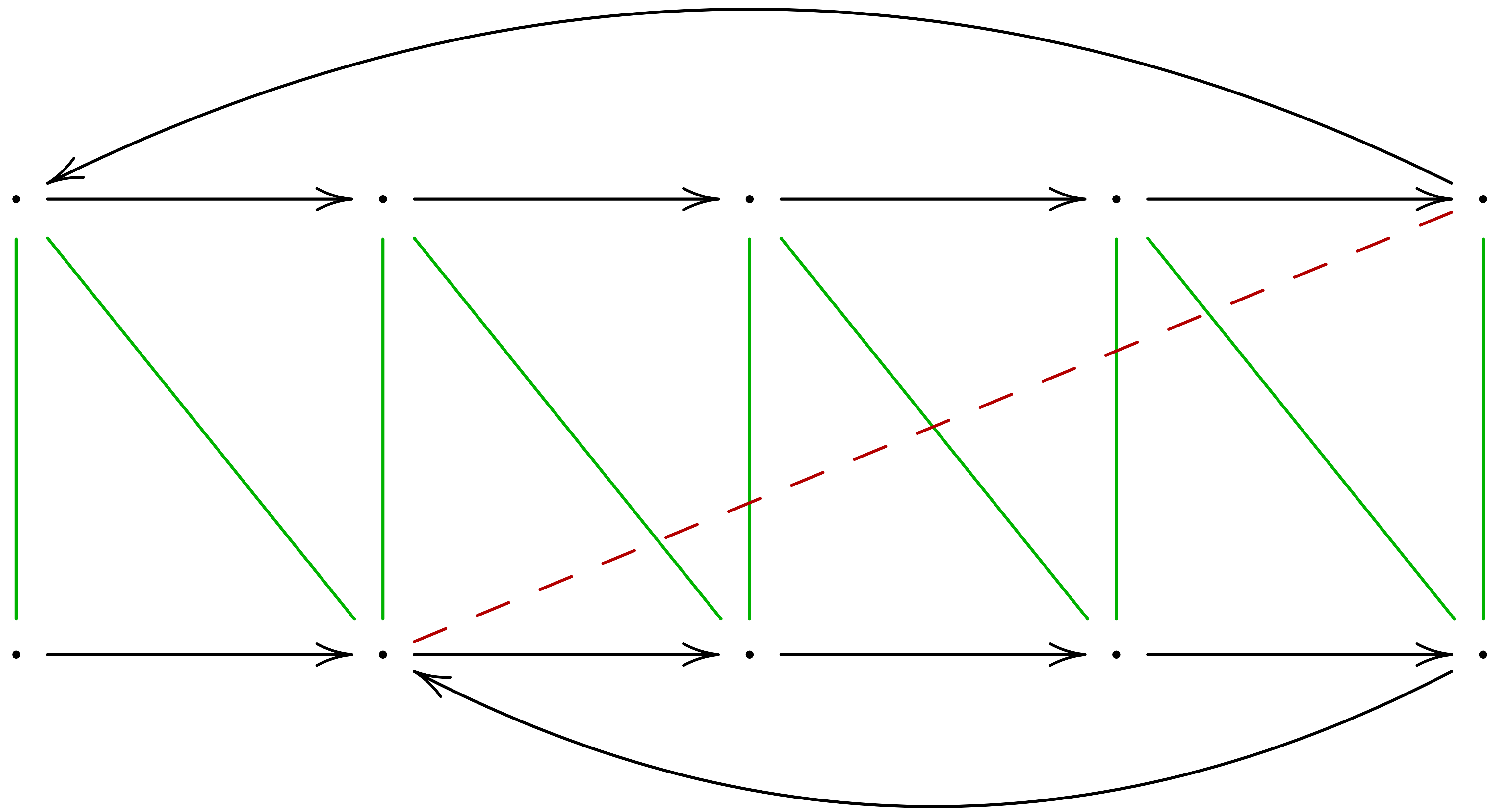
The previous algorithm is quadratic



8 pairs

Complexity

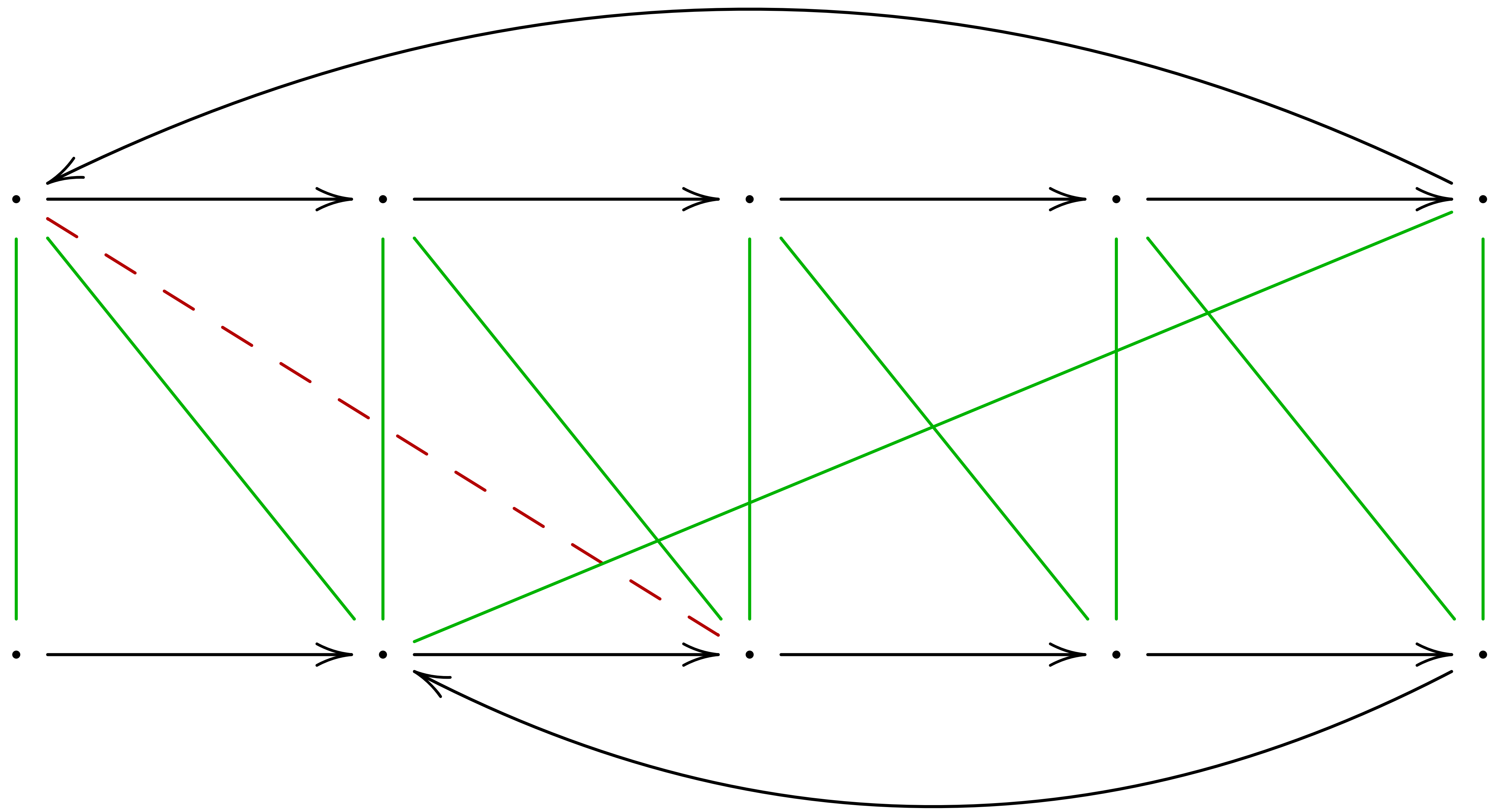
The previous algorithm is quadratic



9 pairs

Complexity

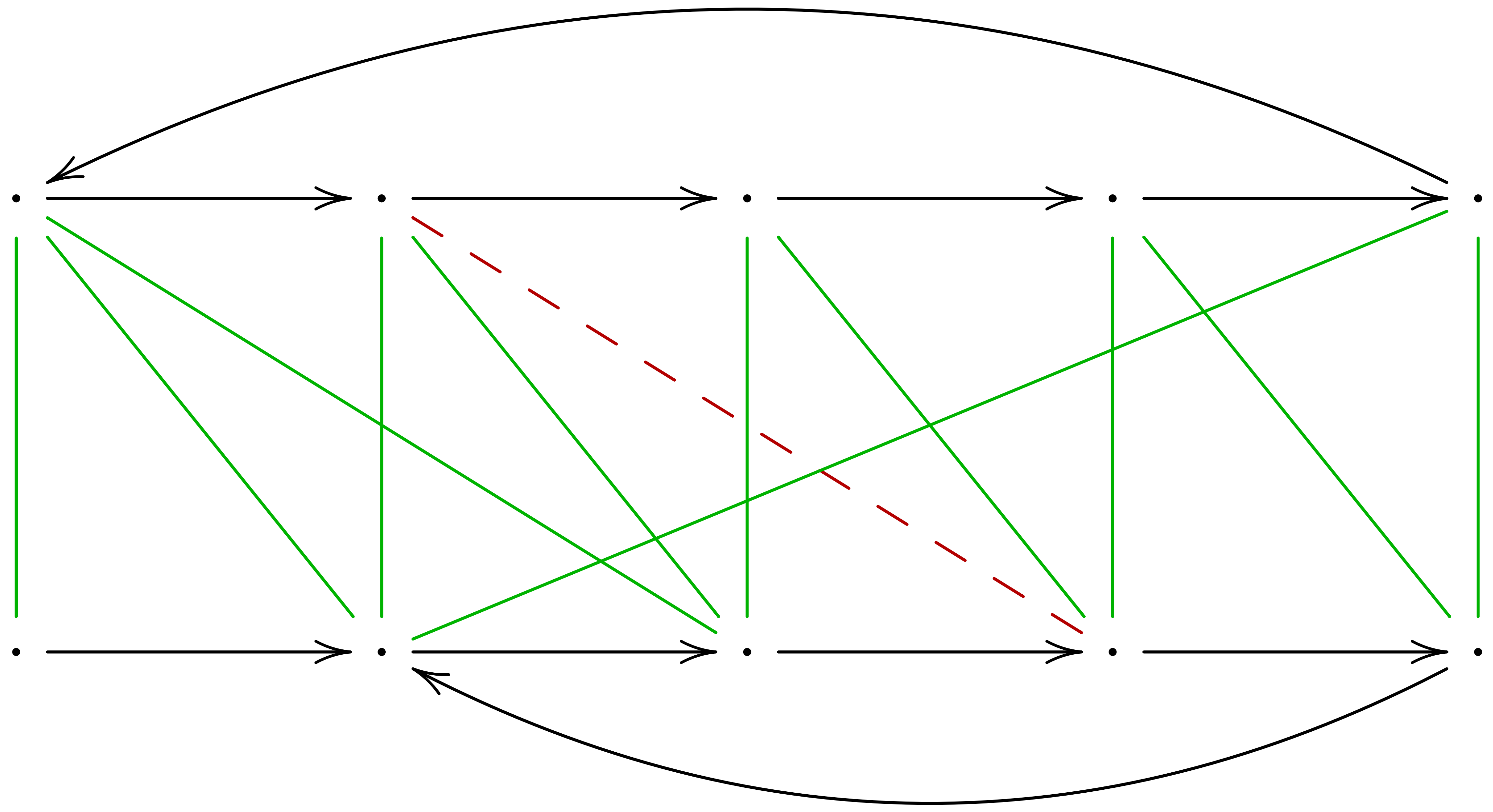
The previous algorithm is quadratic



10 pairs

Complexity

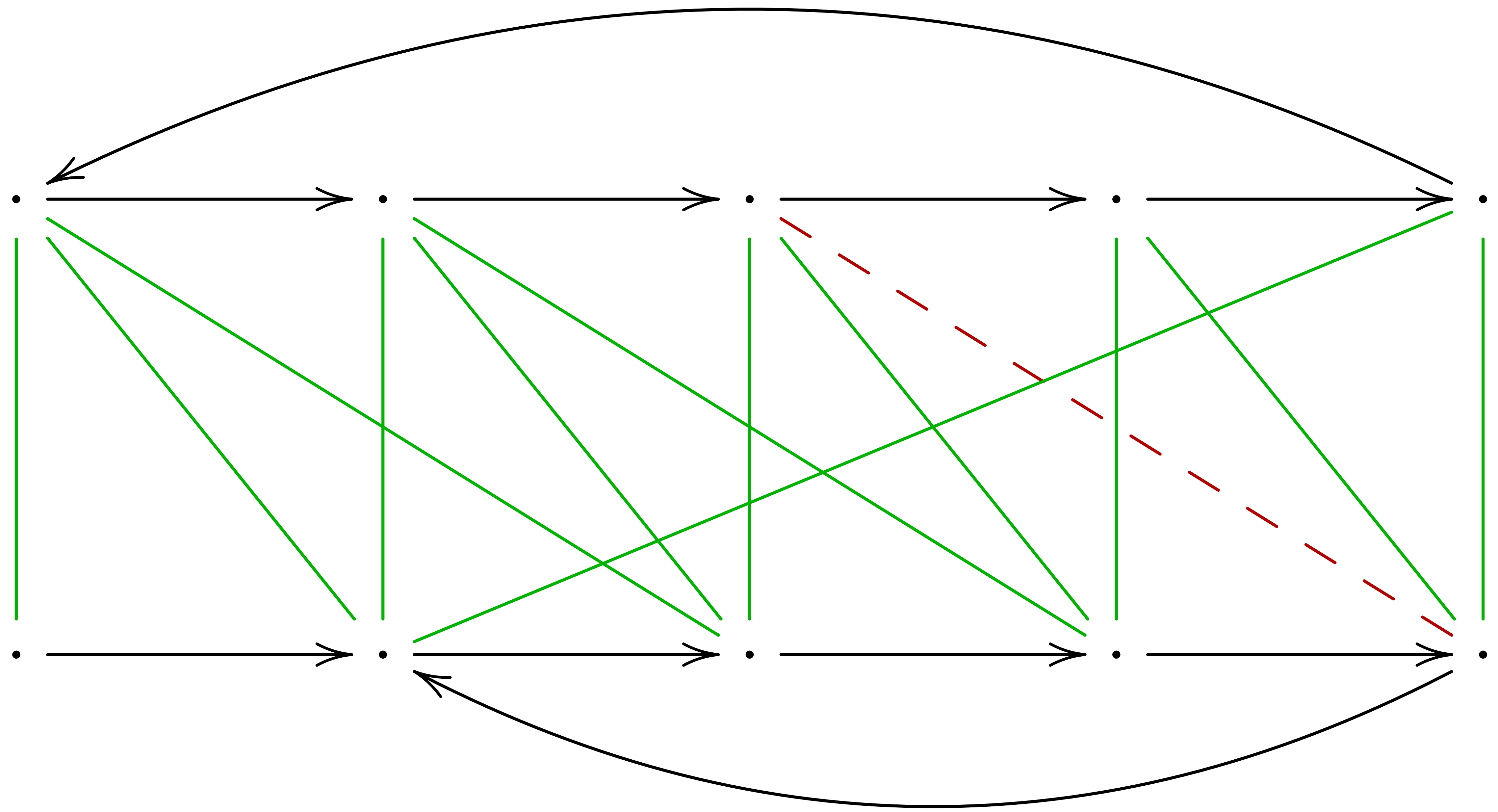
The previous algorithm is quadratic



11 pairs

Complexity

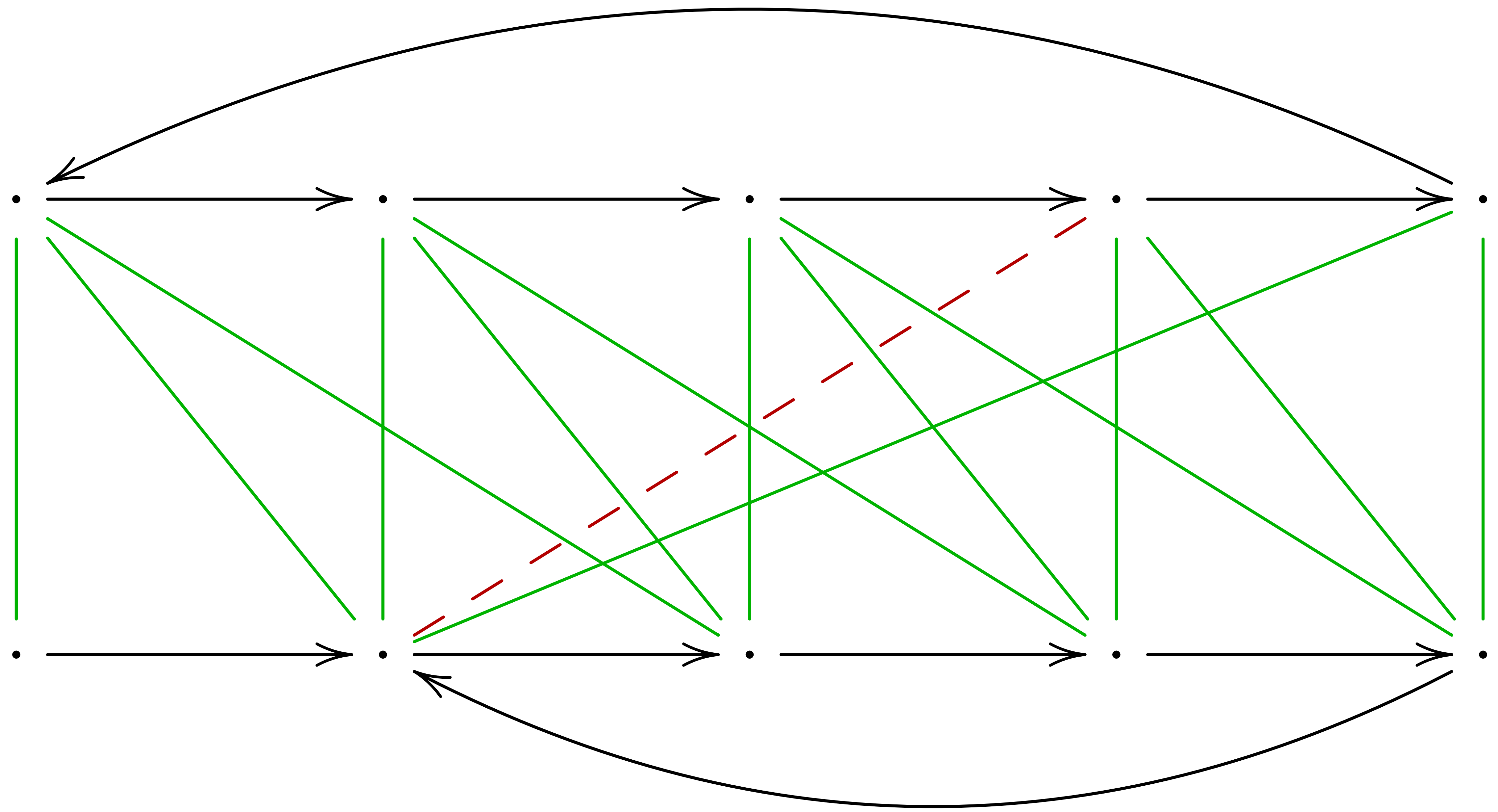
The previous algorithm is quadratic



12 pairs

Complexity

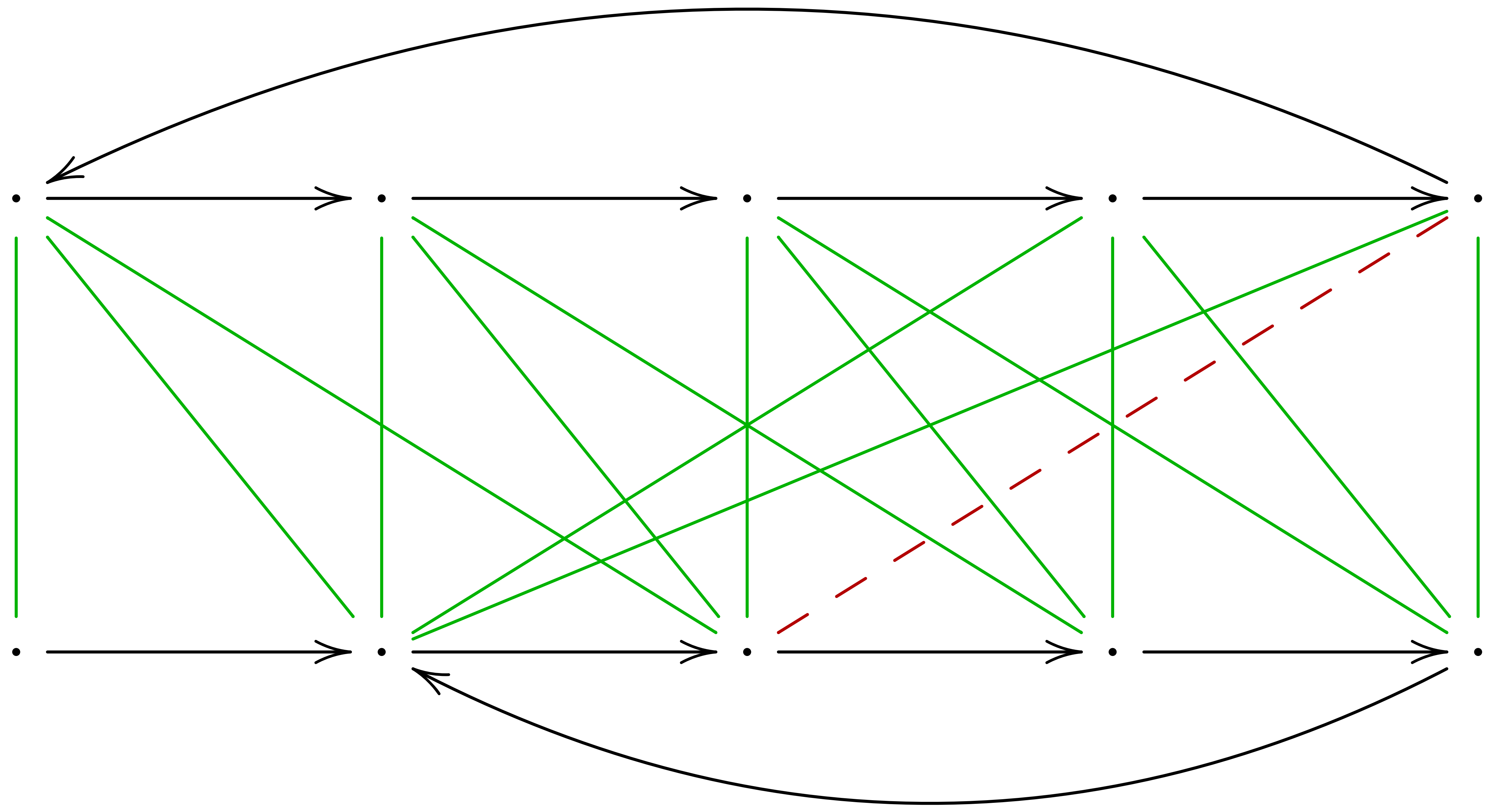
The previous algorithm is quadratic



13 pairs

Complexity

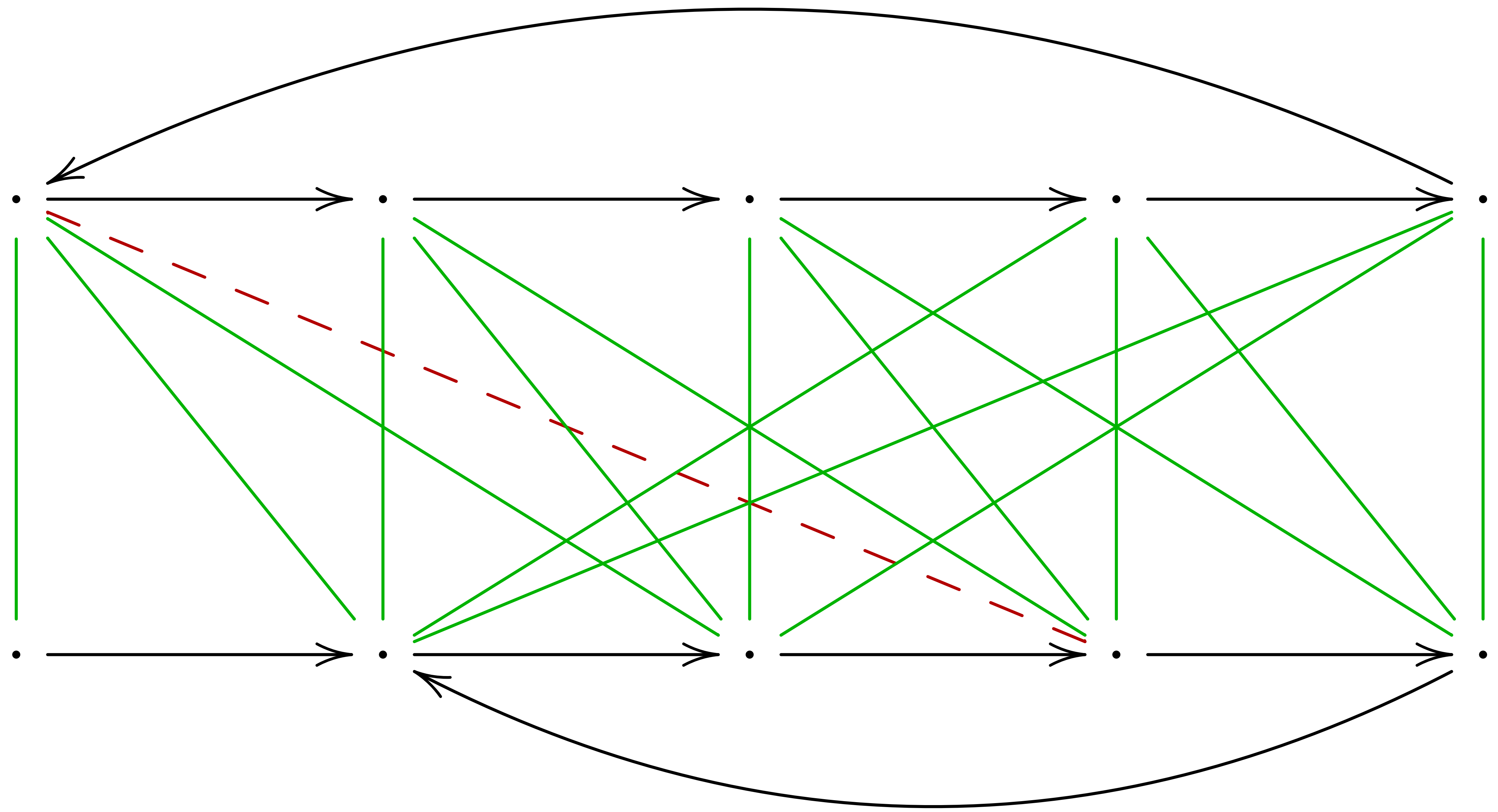
The previous algorithm is quadratic



14 pairs

Complexity

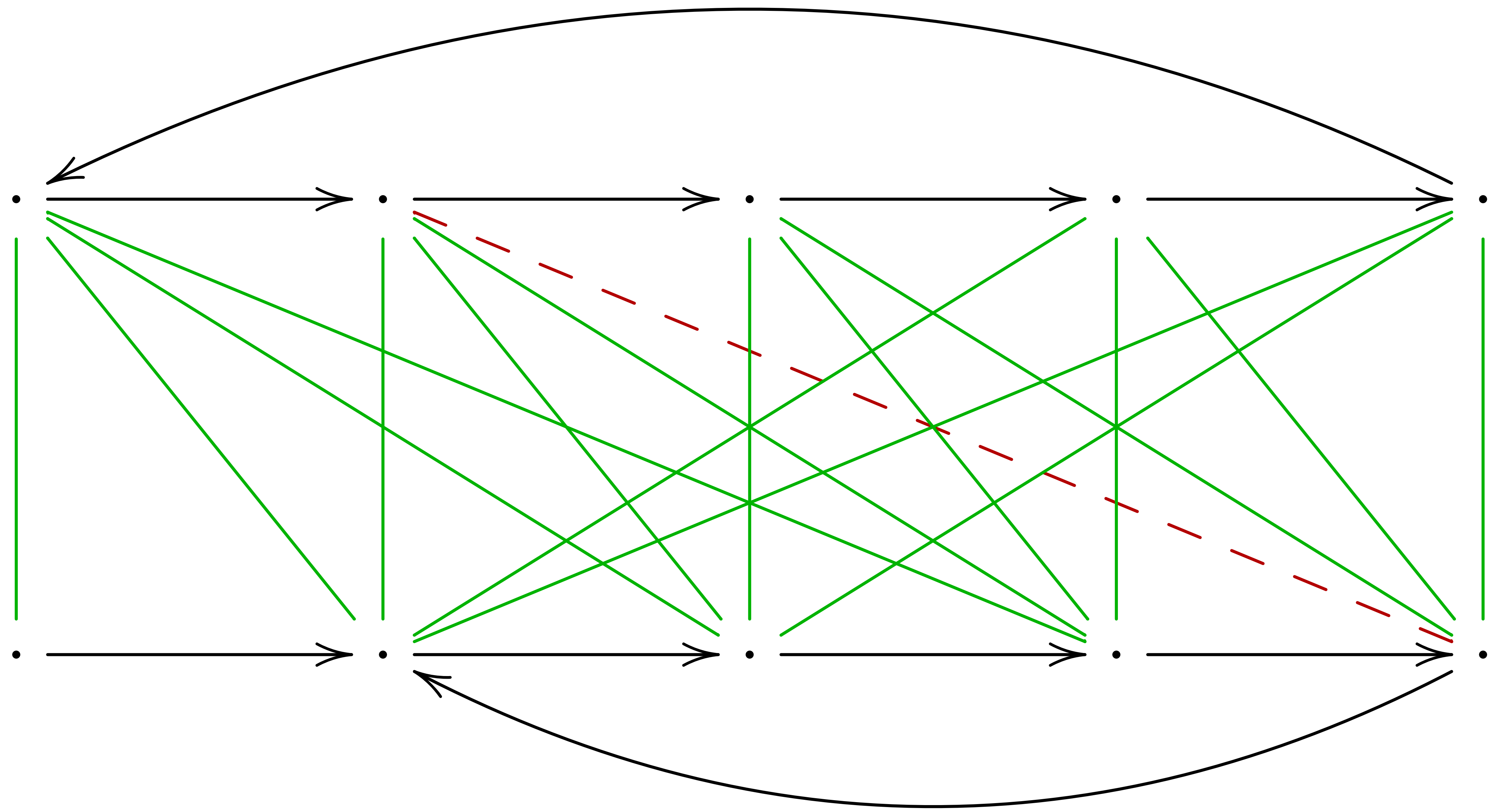
The previous algorithm is quadratic



15 pairs

Complexity

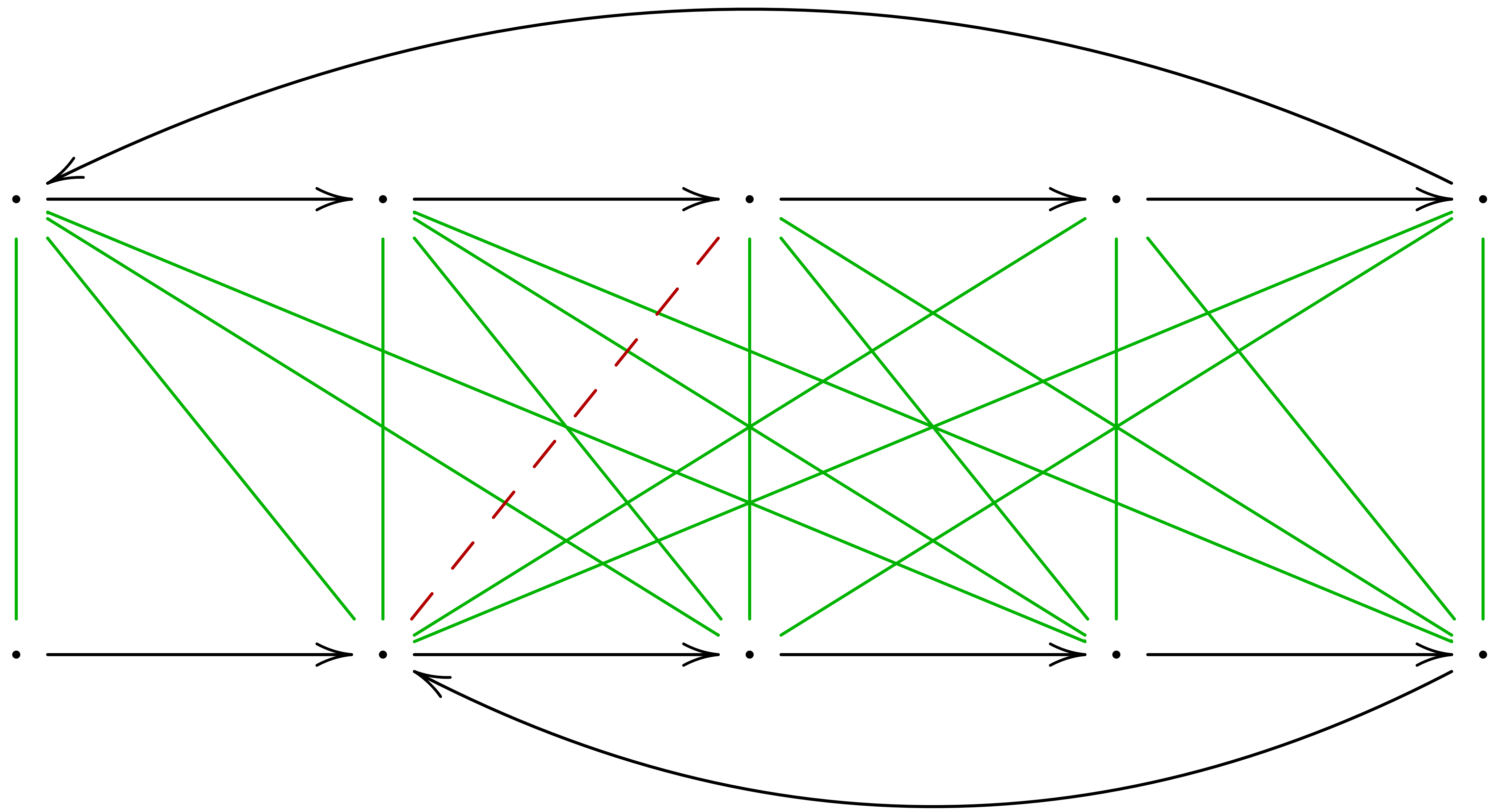
The previous algorithm is quadratic



16 pairs

Complexity

The previous algorithm is quadratic



17 pairs

Complexity

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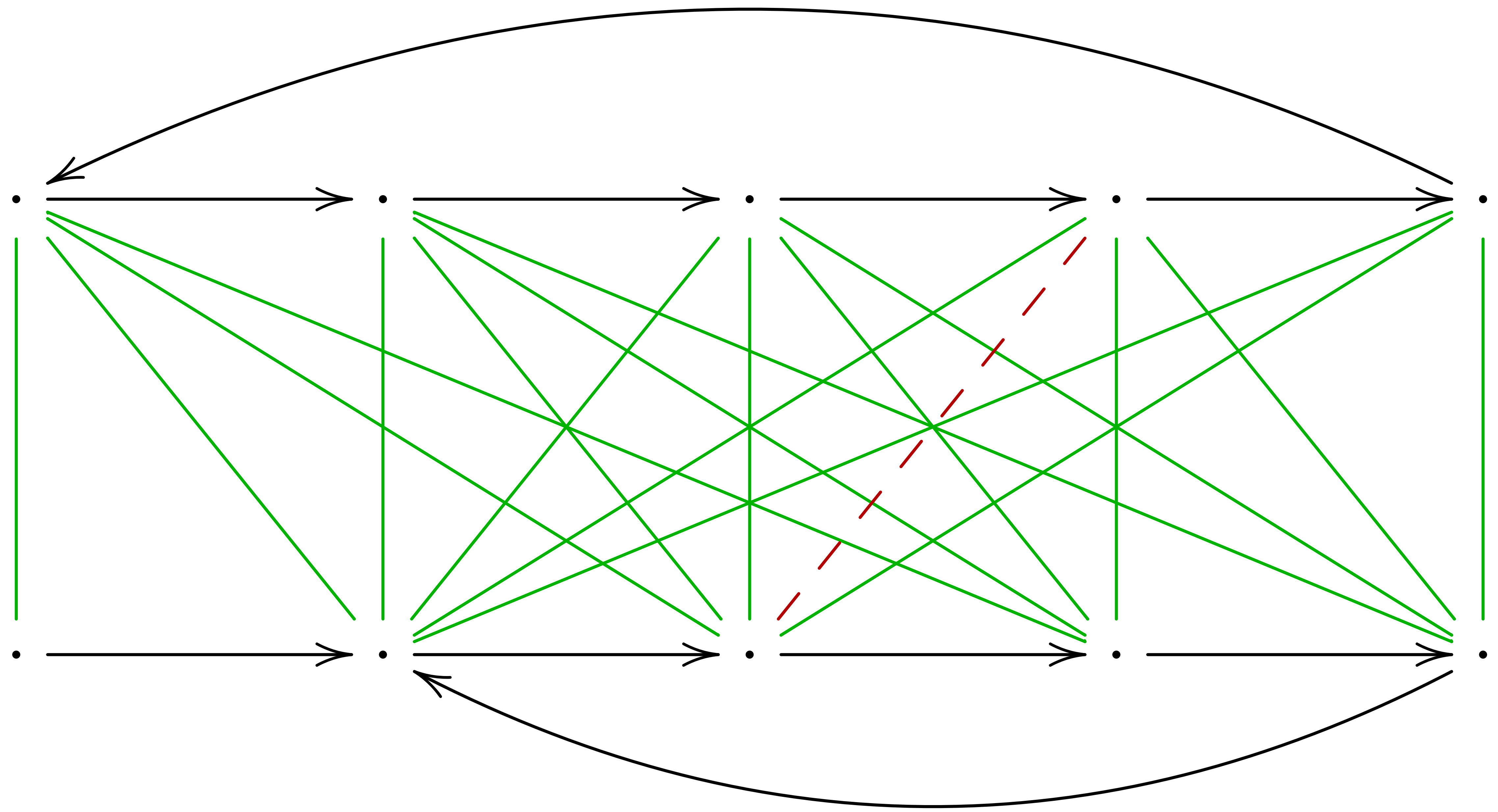

18 pairs

Complexity

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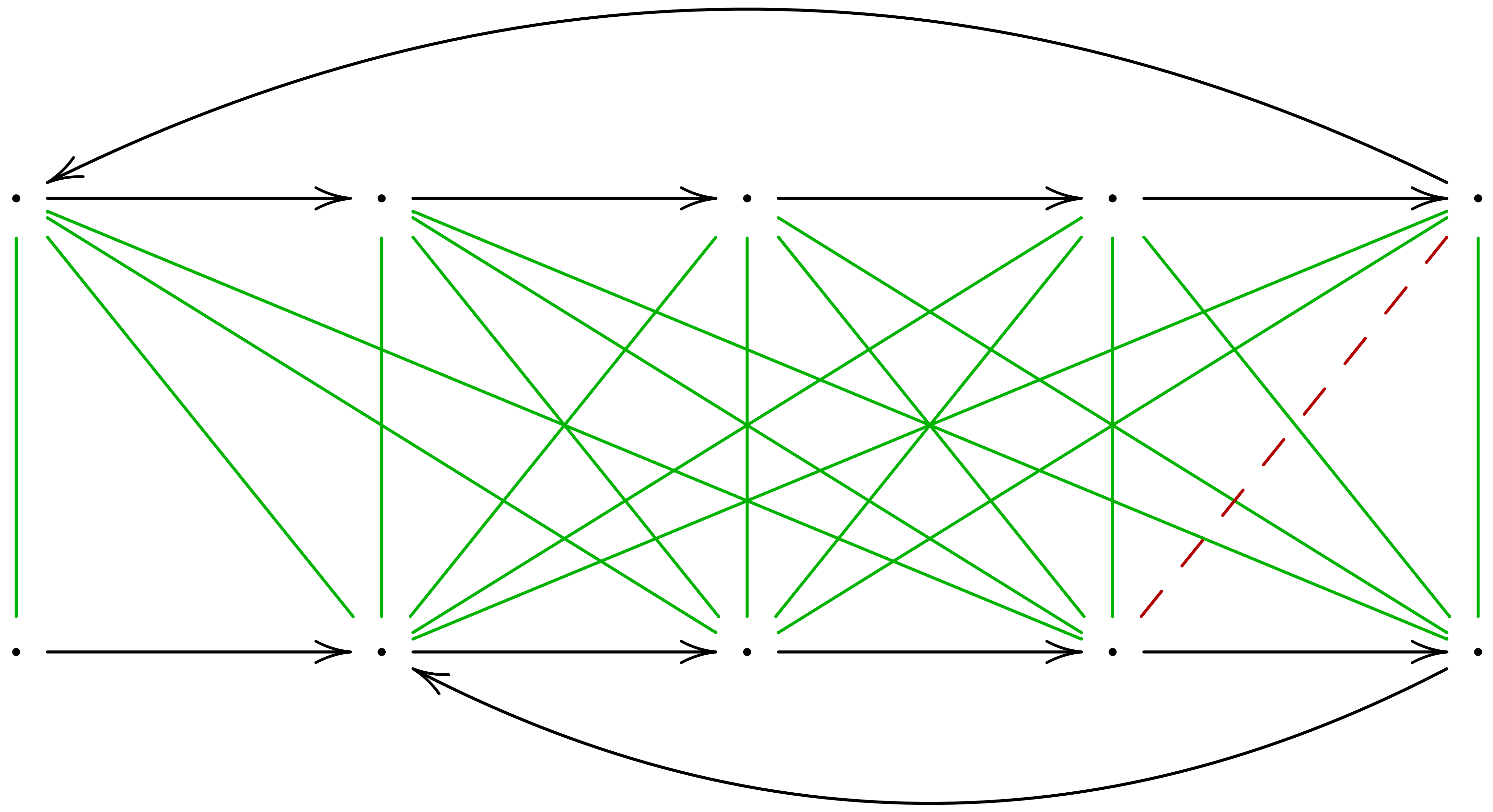
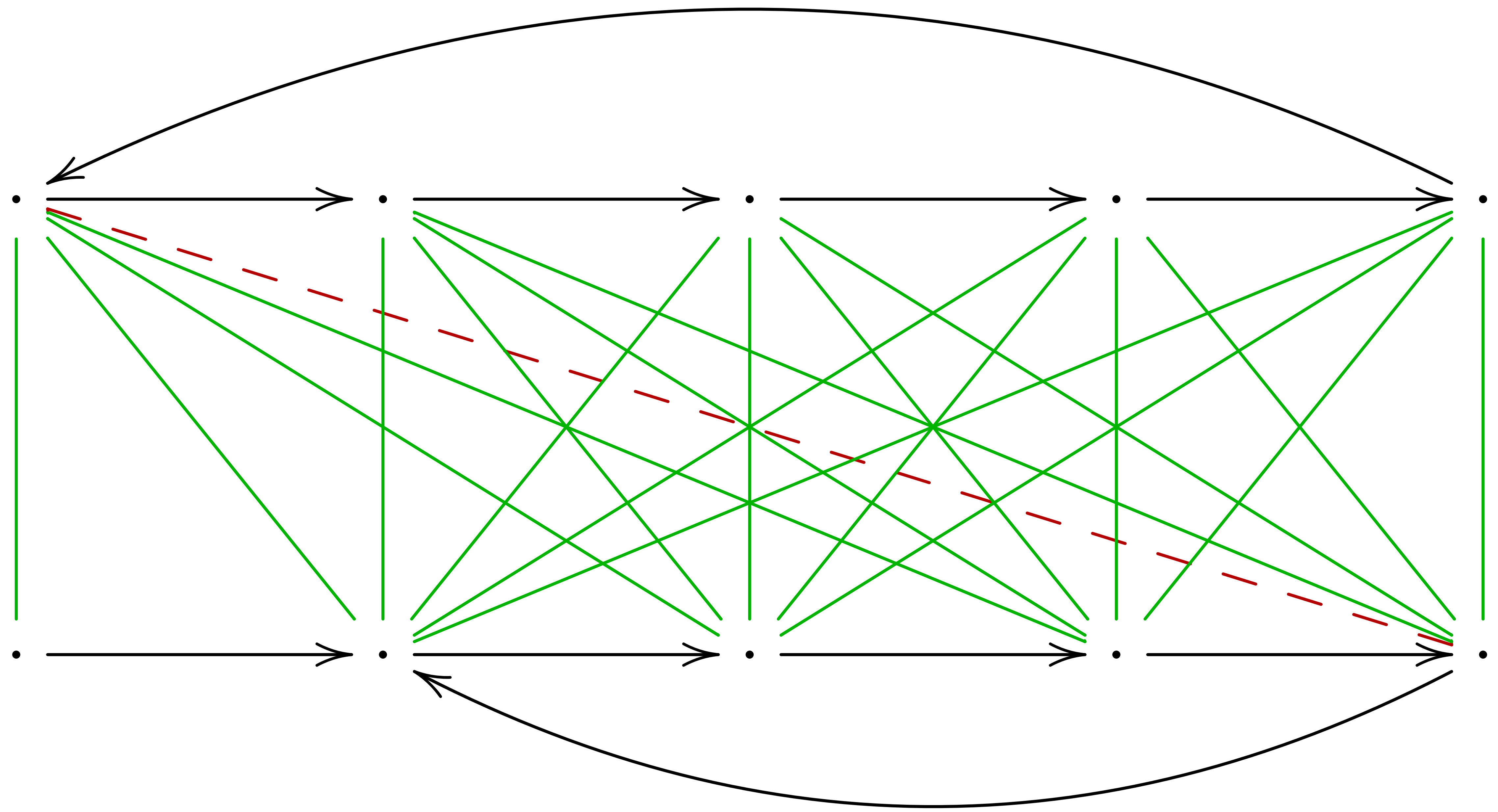


19 pairs

Complexity

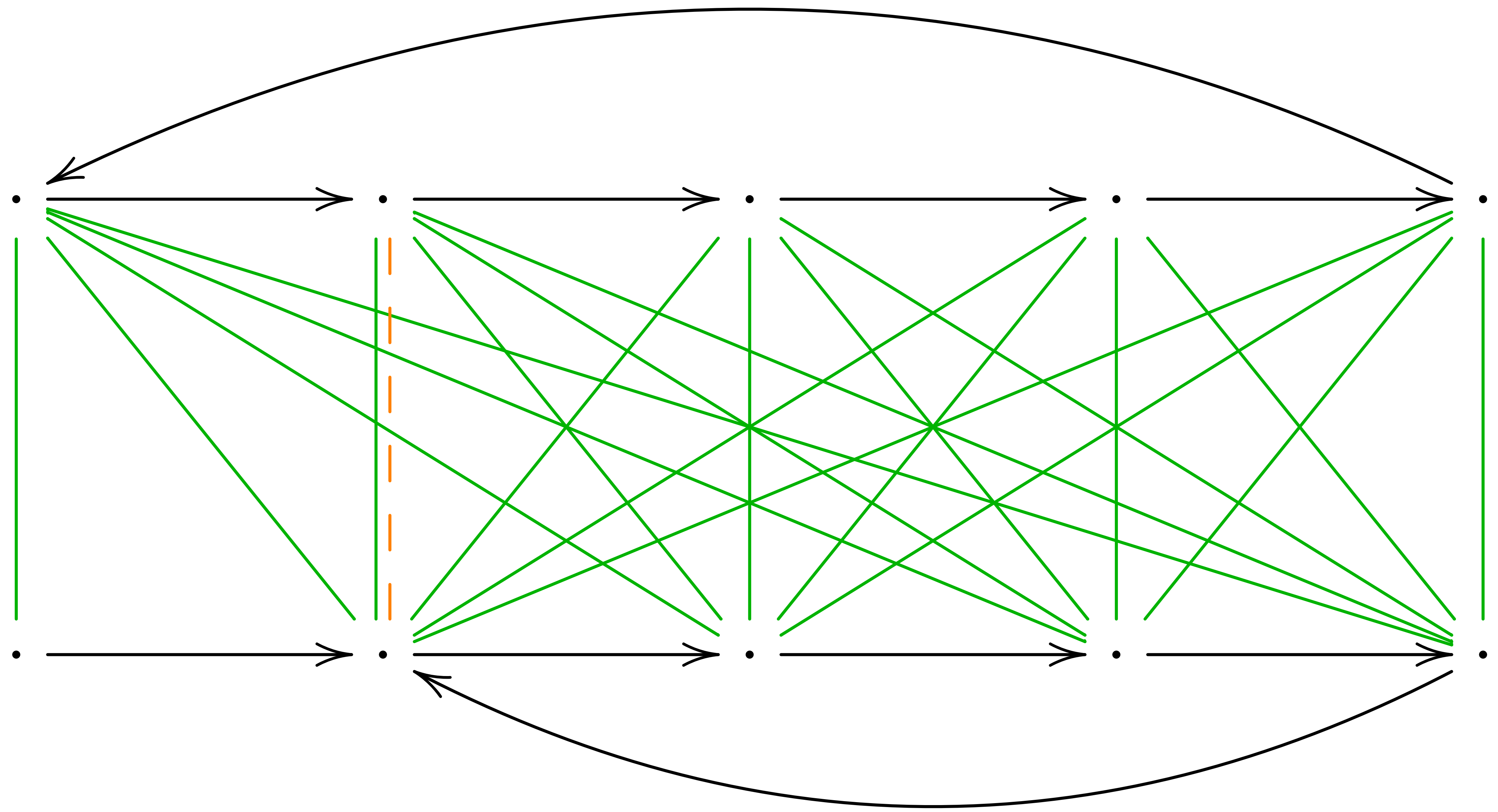
The previous algorithm is quadratic



20 pairs

Complexity

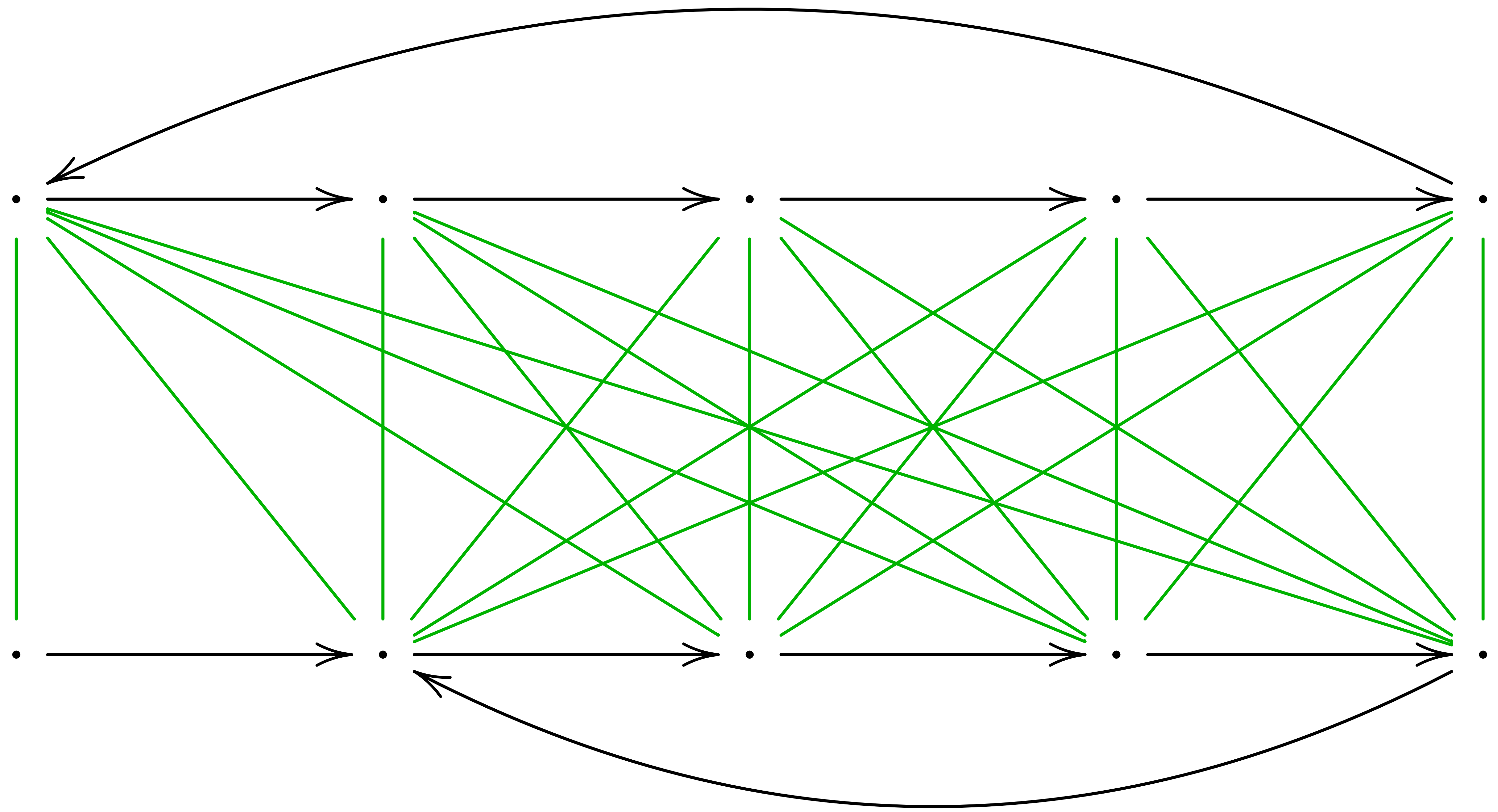
The previous algorithm is quadratic



21 pairs

Complexity

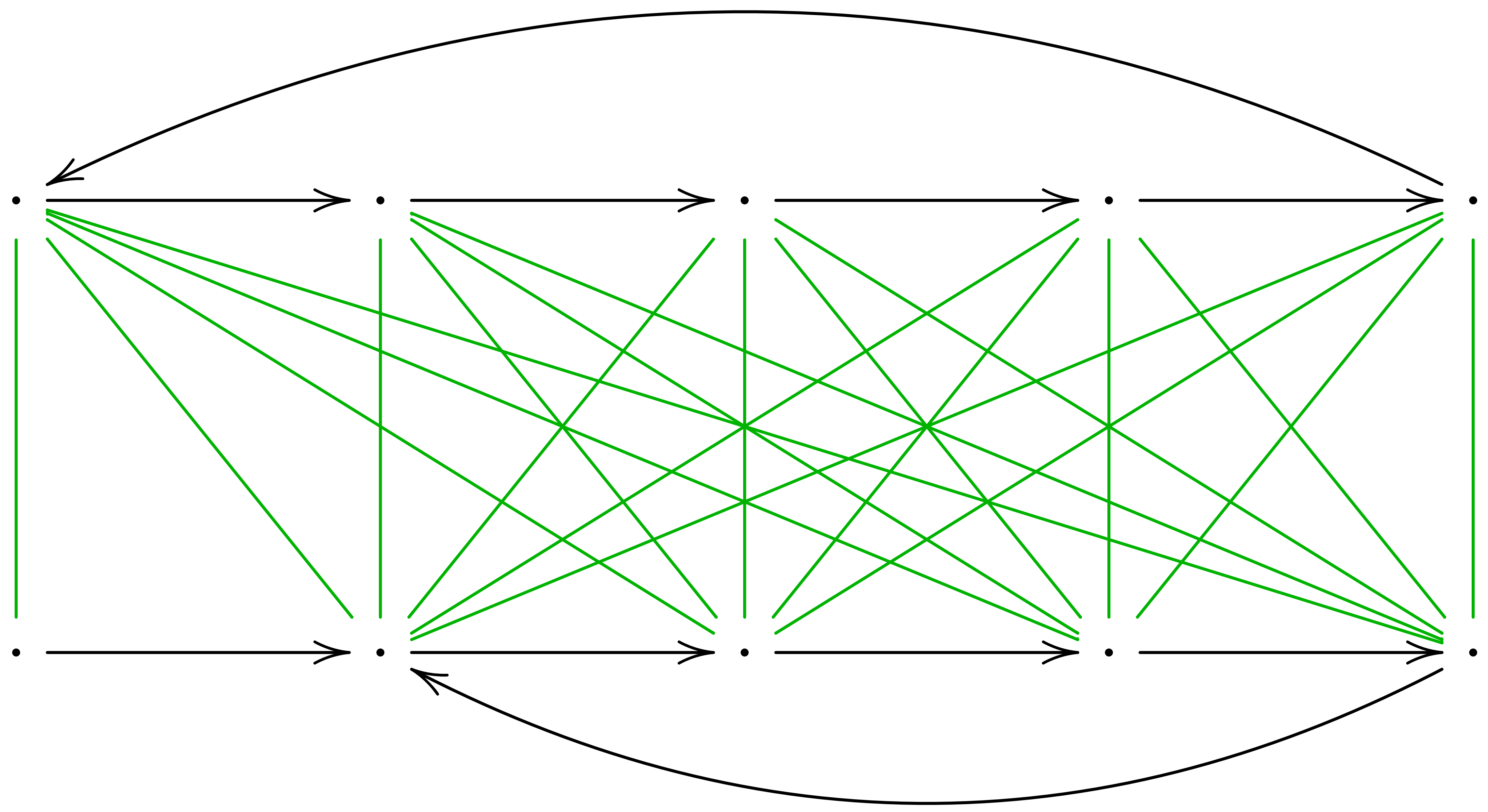
The previous algorithm is quadratic



21 pairs

First improvement

One can stop much earlier



21 pairs

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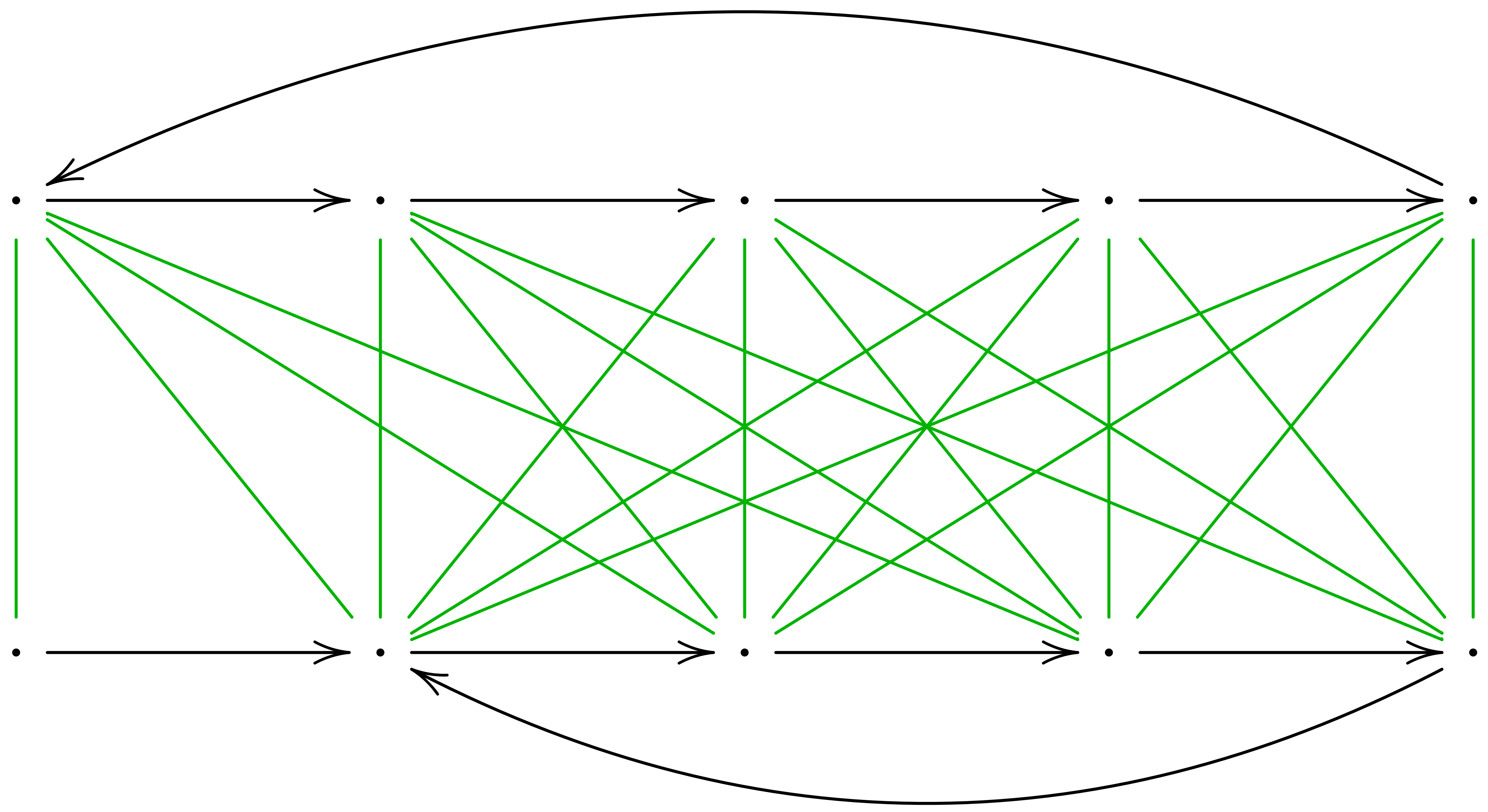

~~21~~ 20 pairs

First improvement

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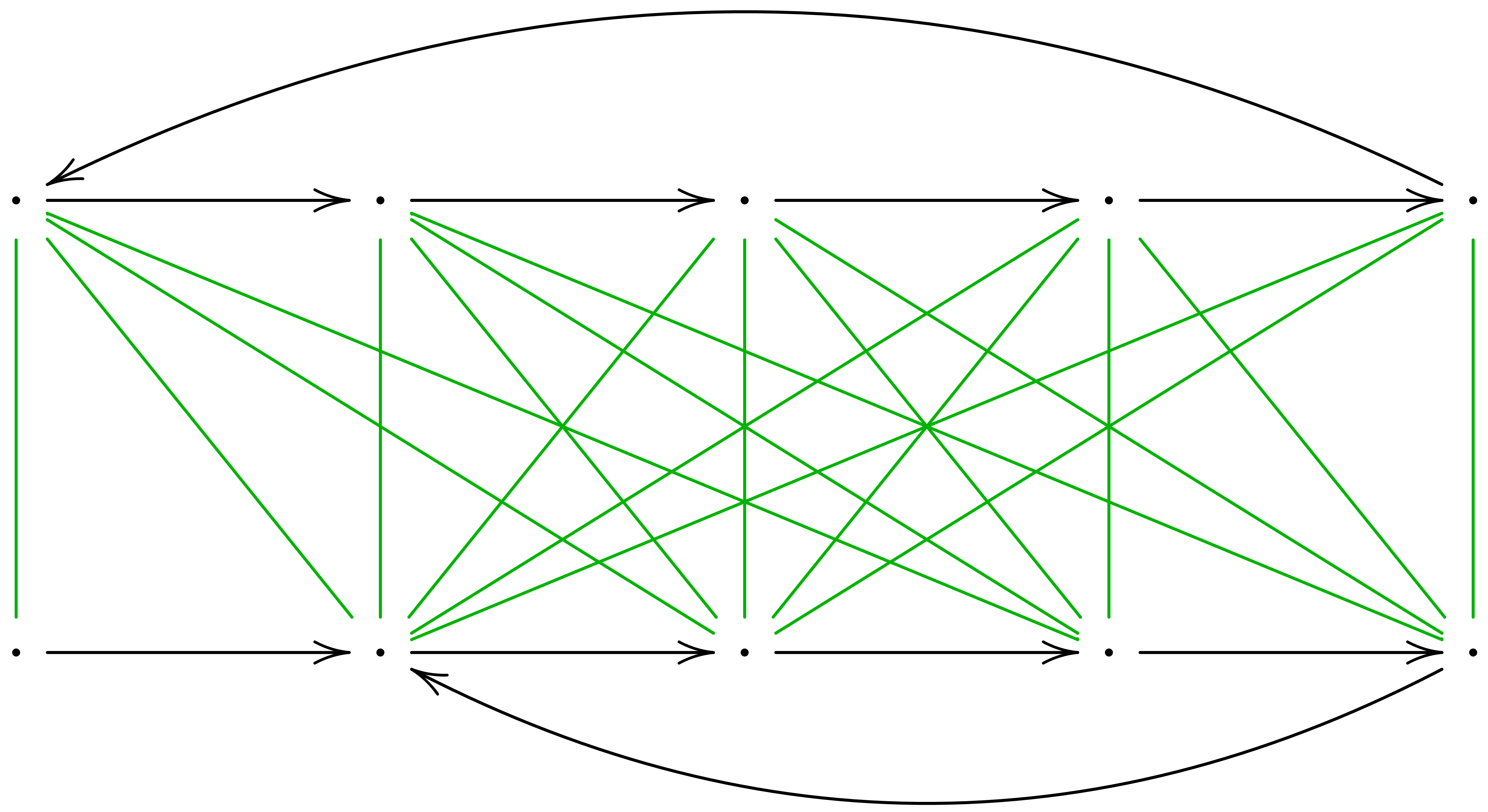

~~21~~ 19 pairs

First improvement

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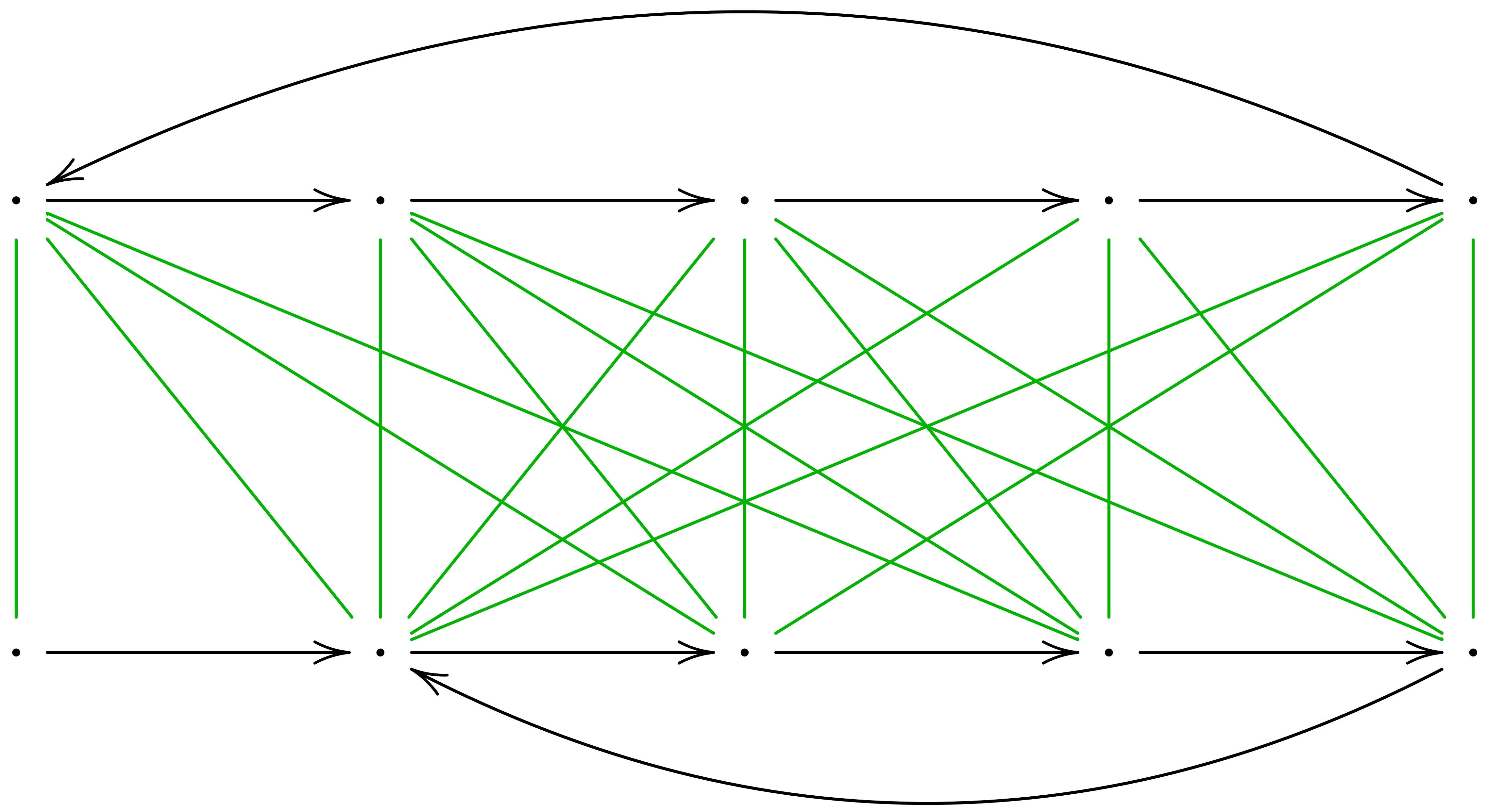

~~21~~ 18 pairs

First improvement

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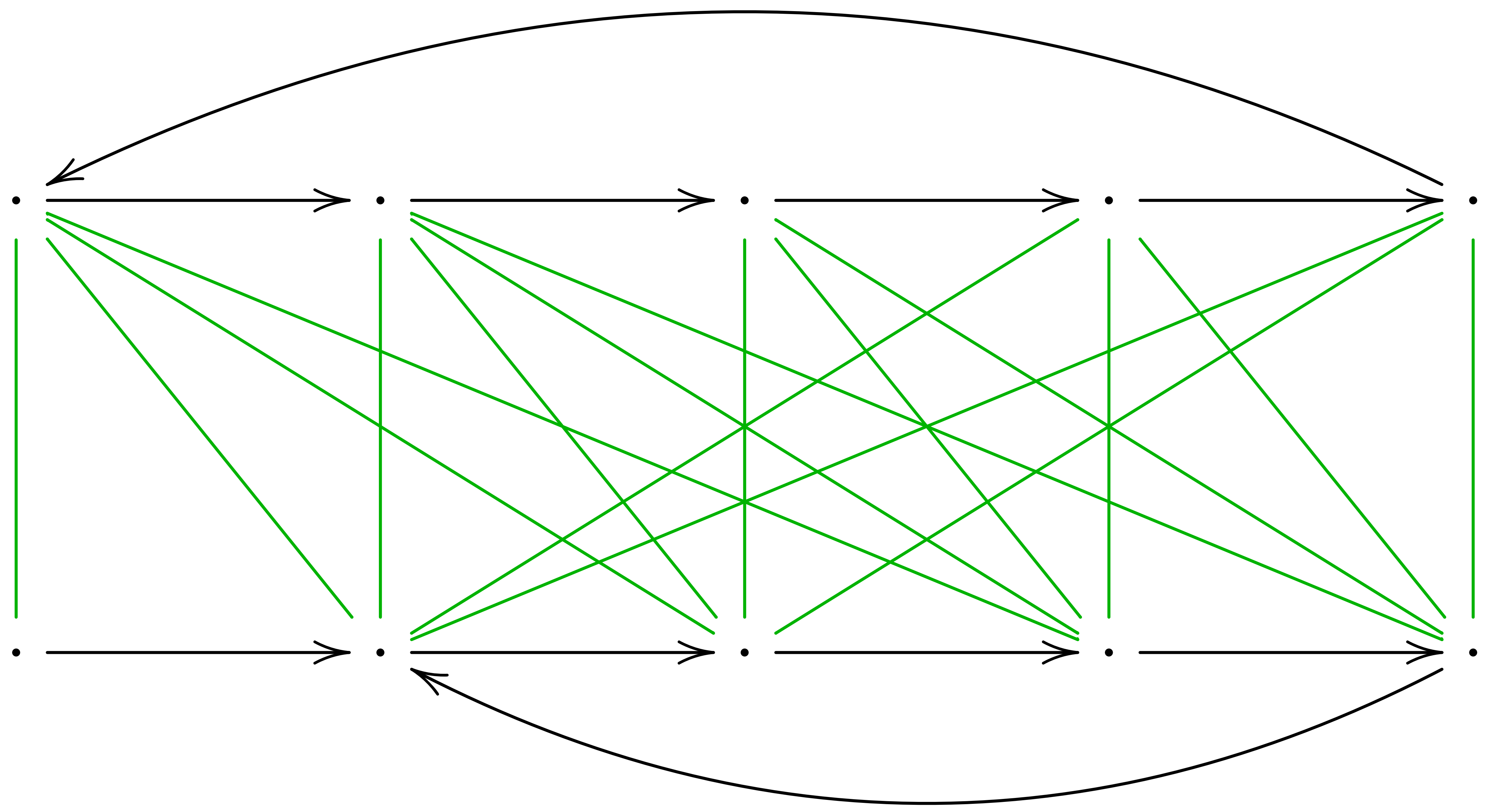

~~21~~ 17 pairs

First improvement

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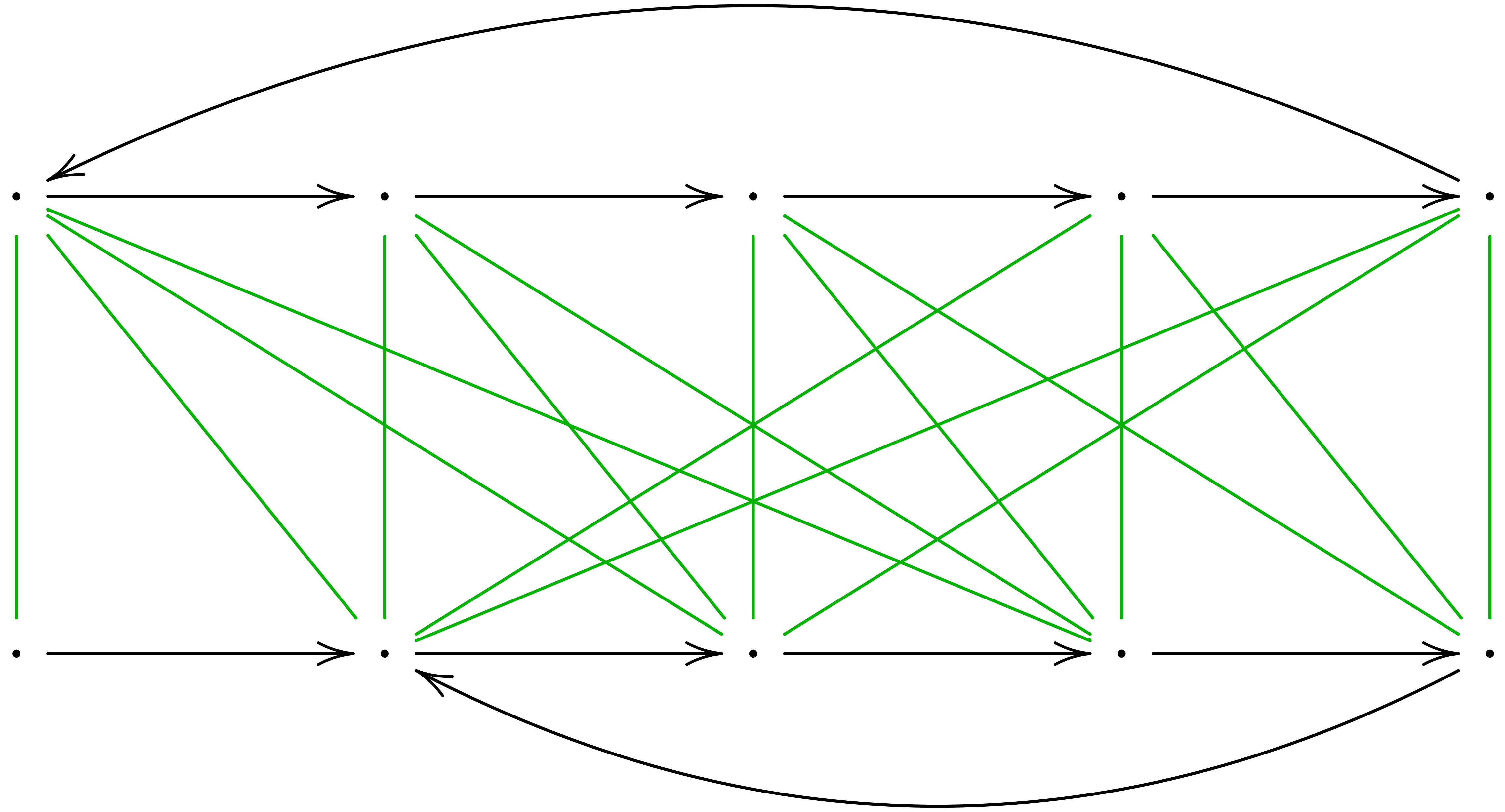

~~21~~ 16 pairs

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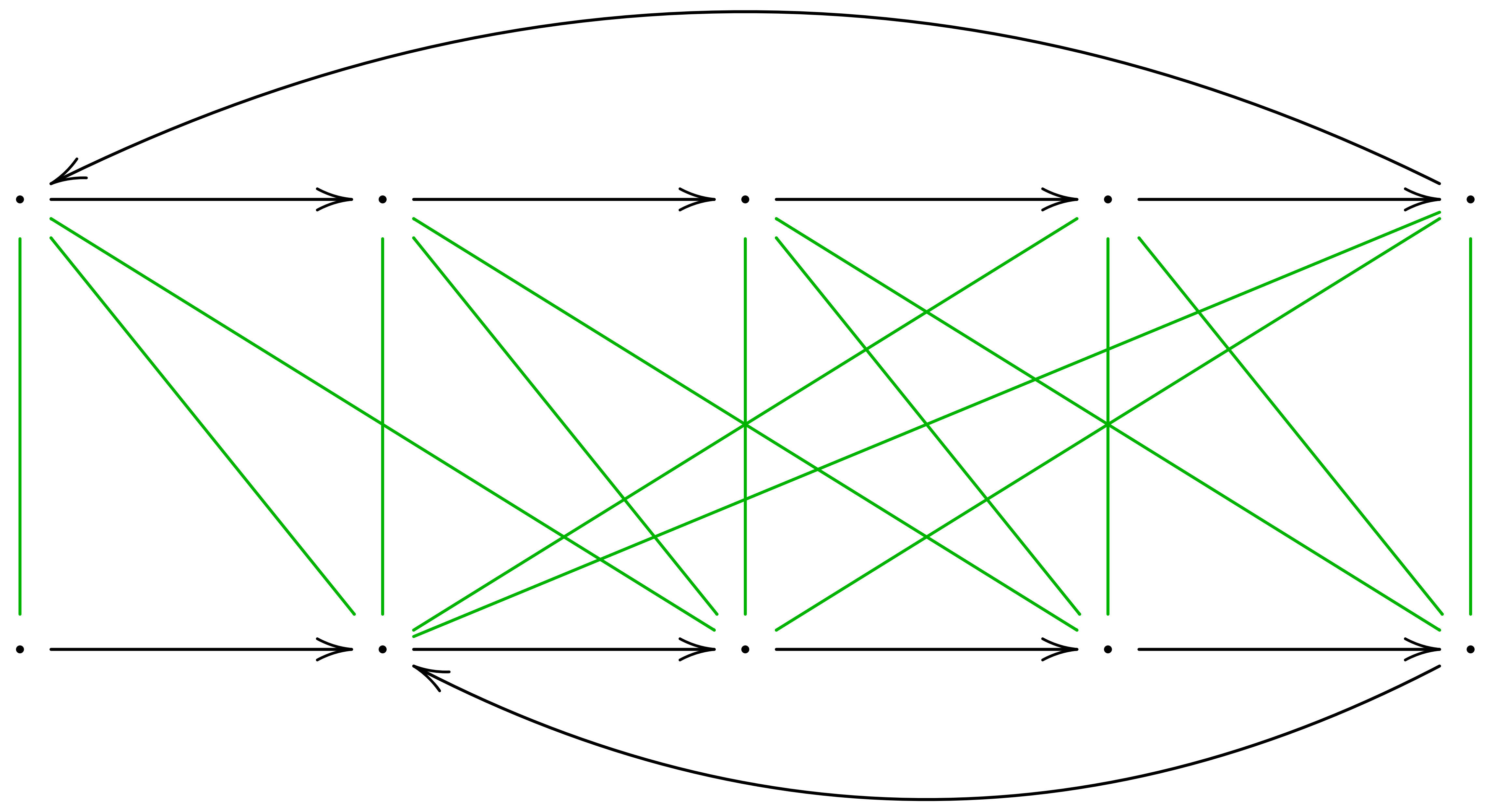

~~21~~ 15 pairs

First improvement

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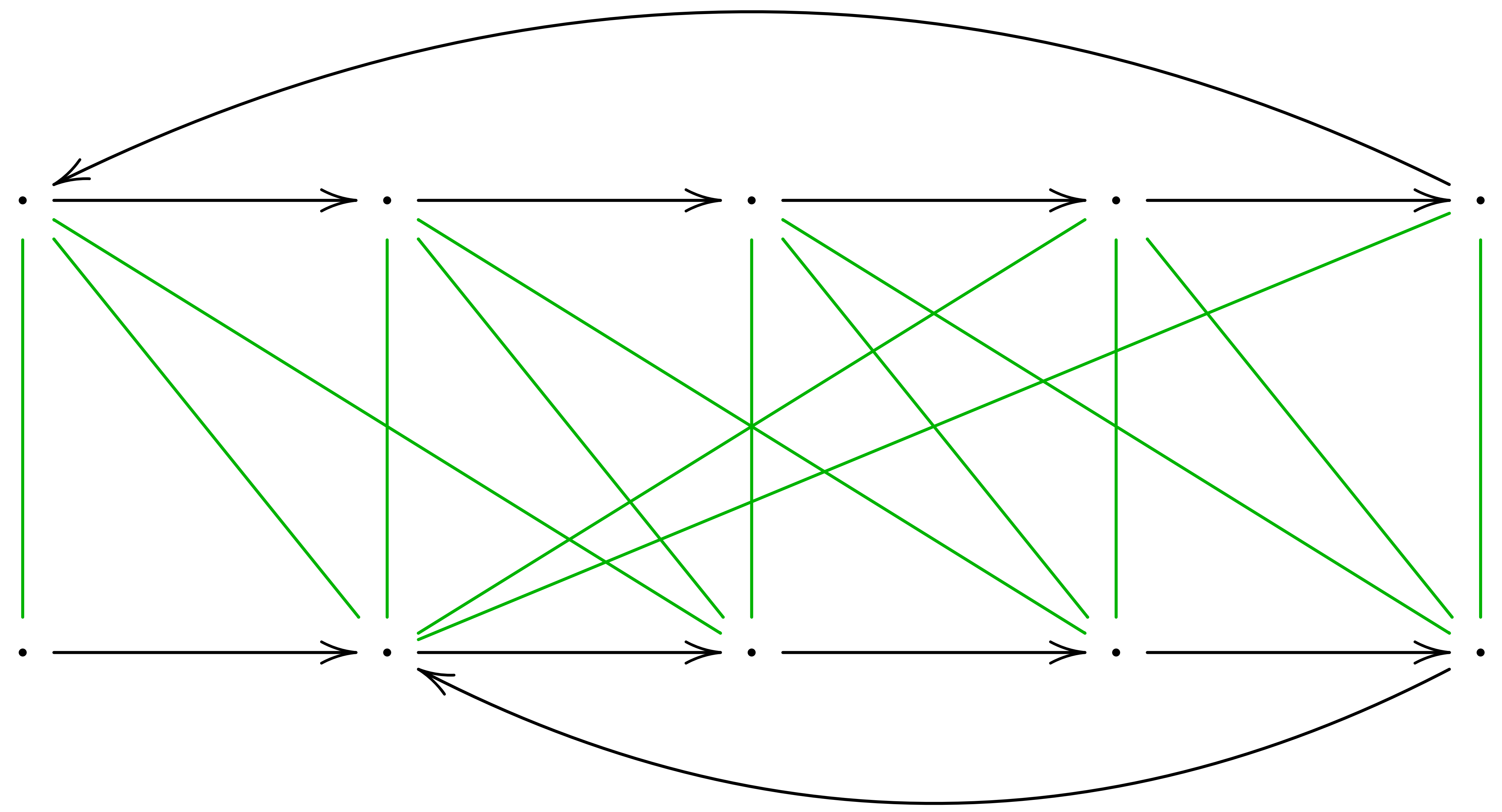

~~21~~ 14 pairs

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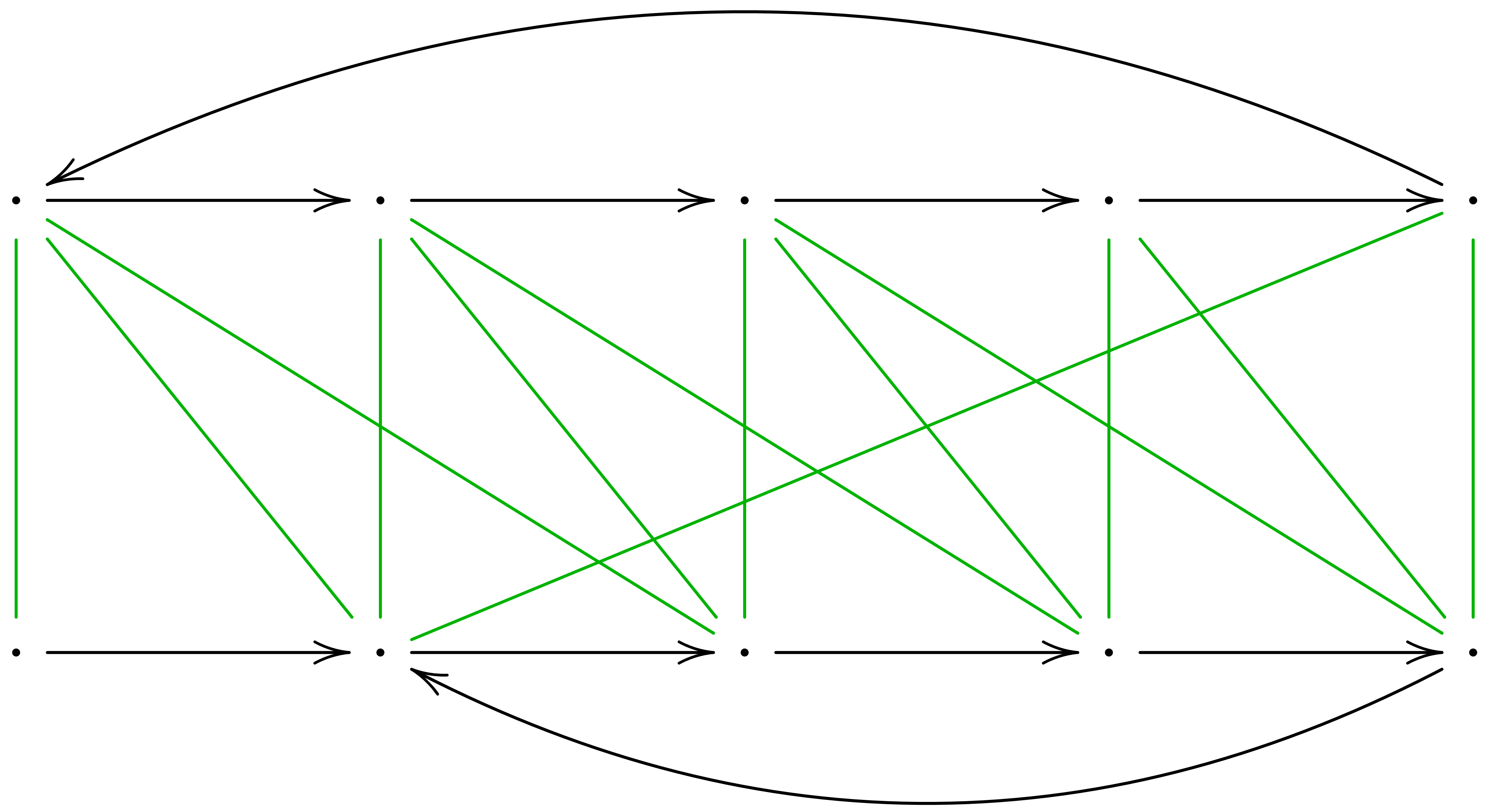

~~21~~ 13 pairs

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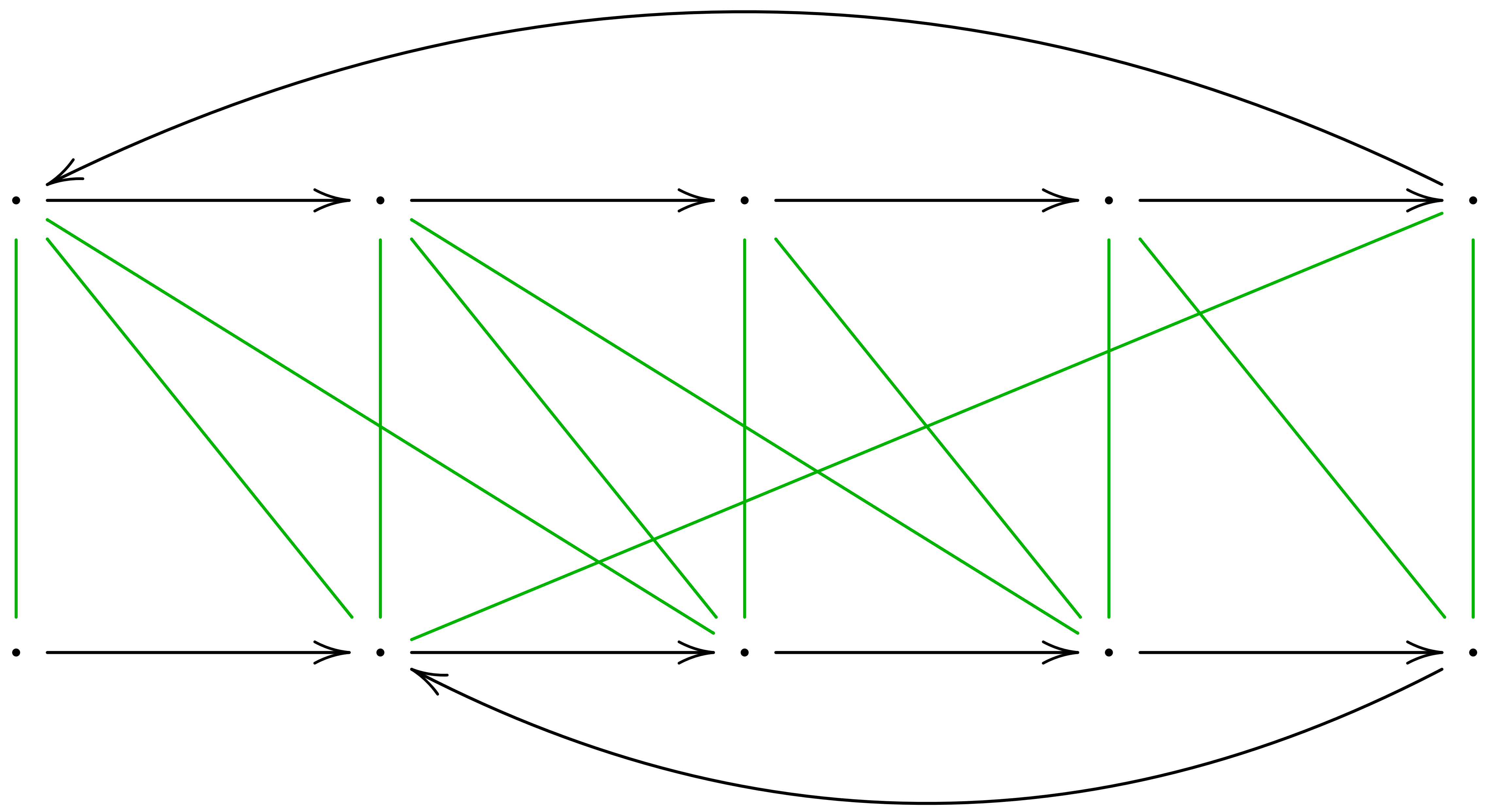

~~21~~ 12 pairs

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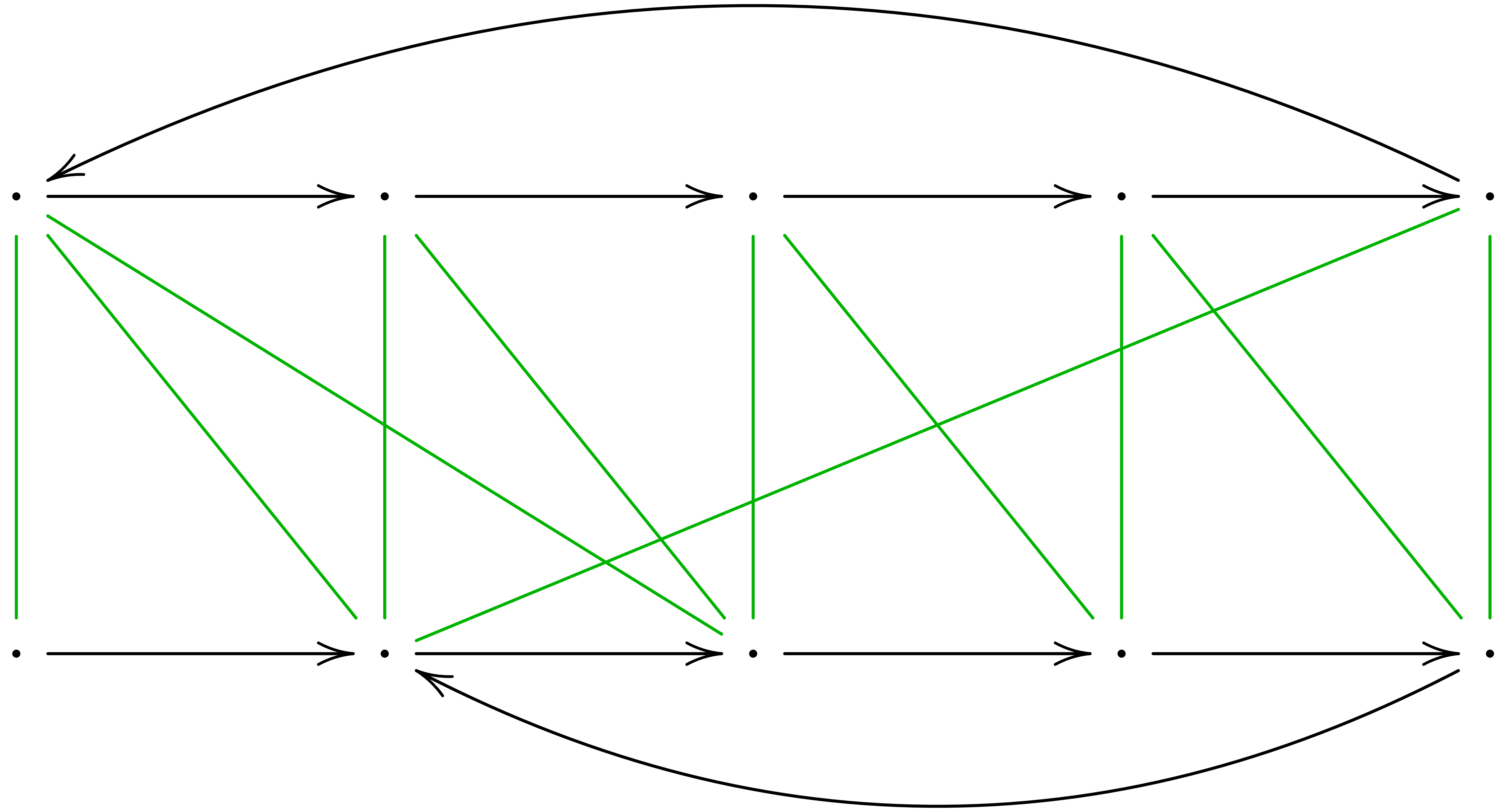

~~21~~ 11 pairs

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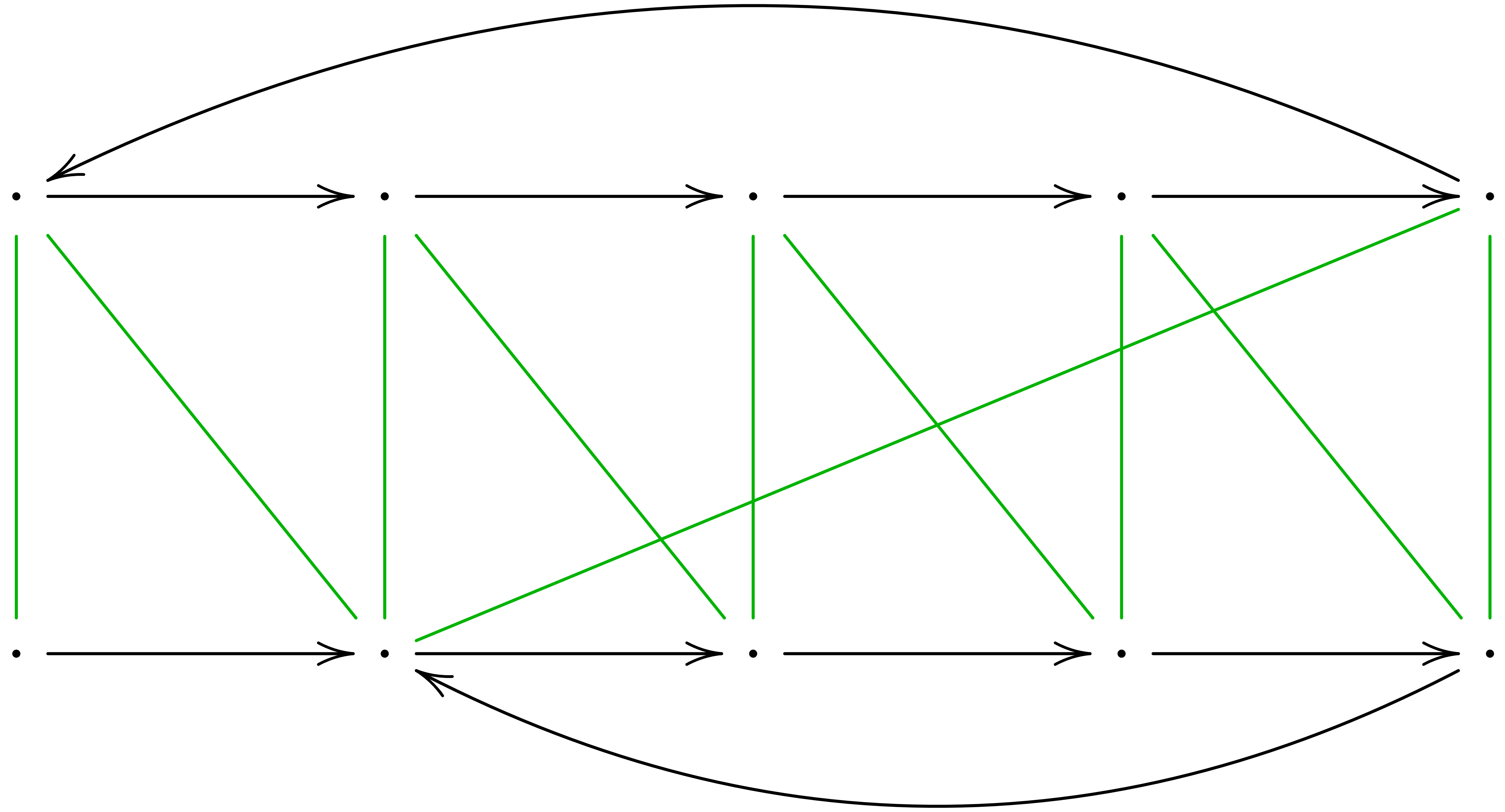

~~21~~ 10 pairs

First improvement

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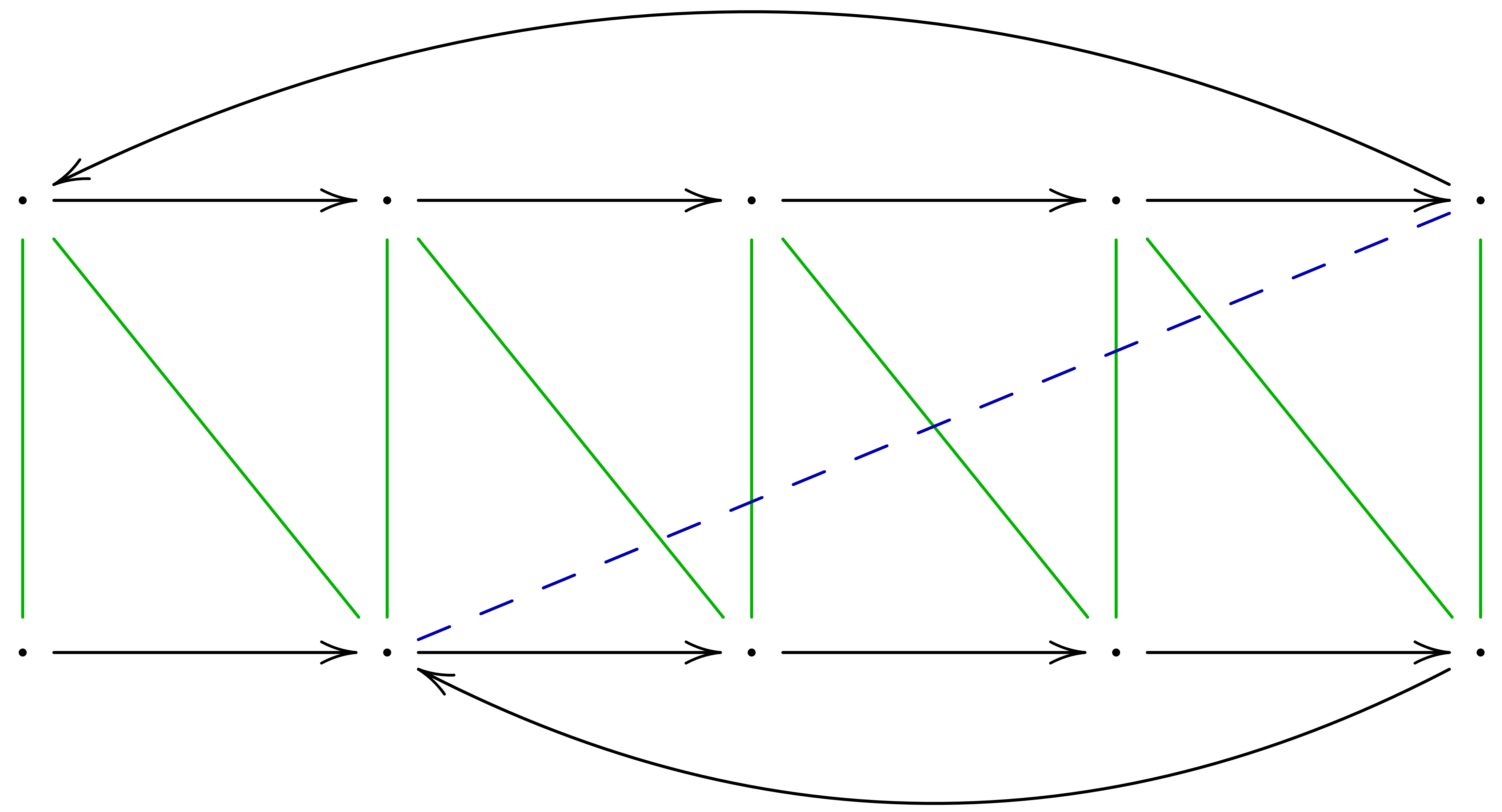

~~21~~ 9 pairs

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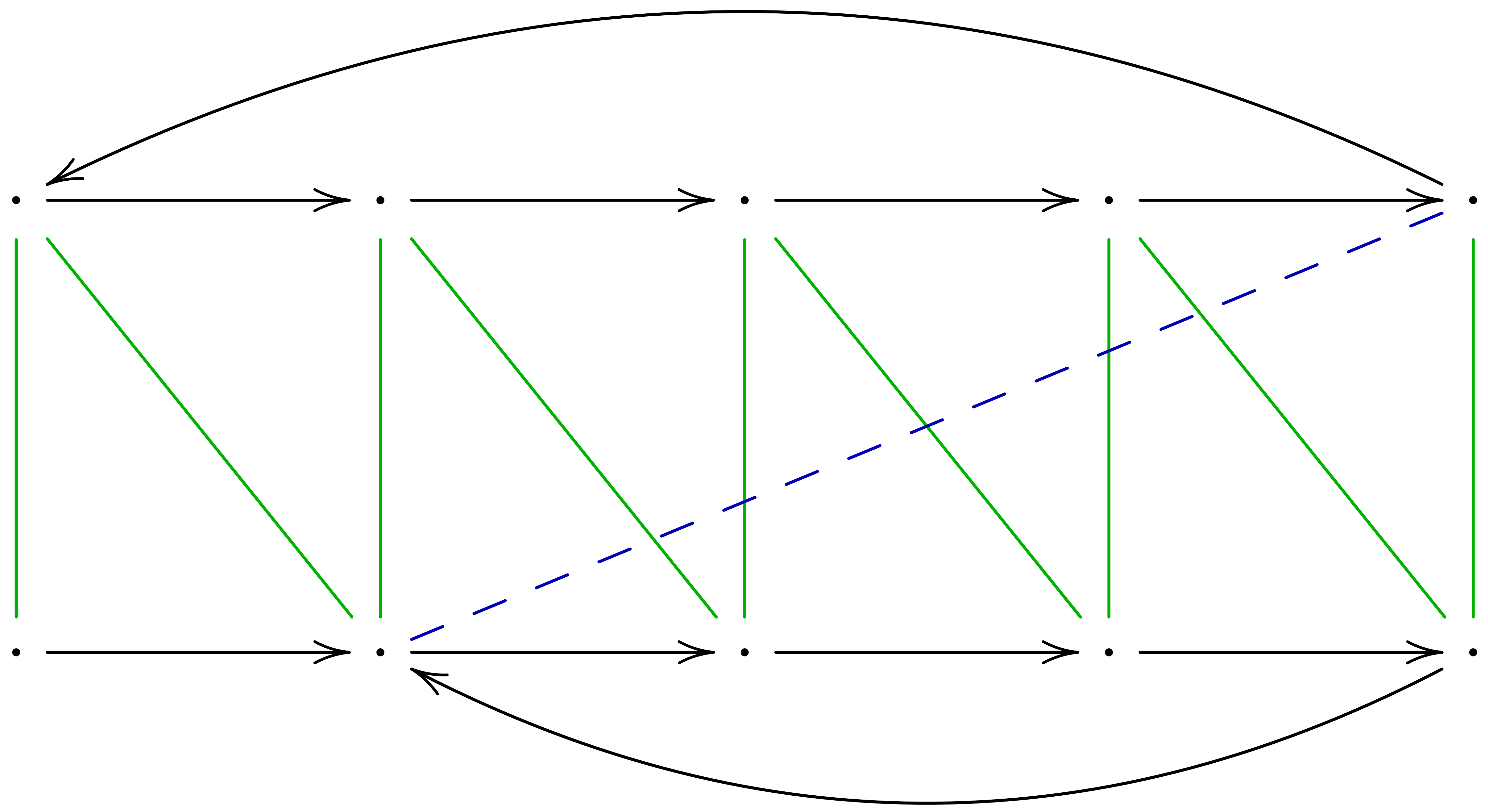

[Hopcroft and Karp '71]

First improvement

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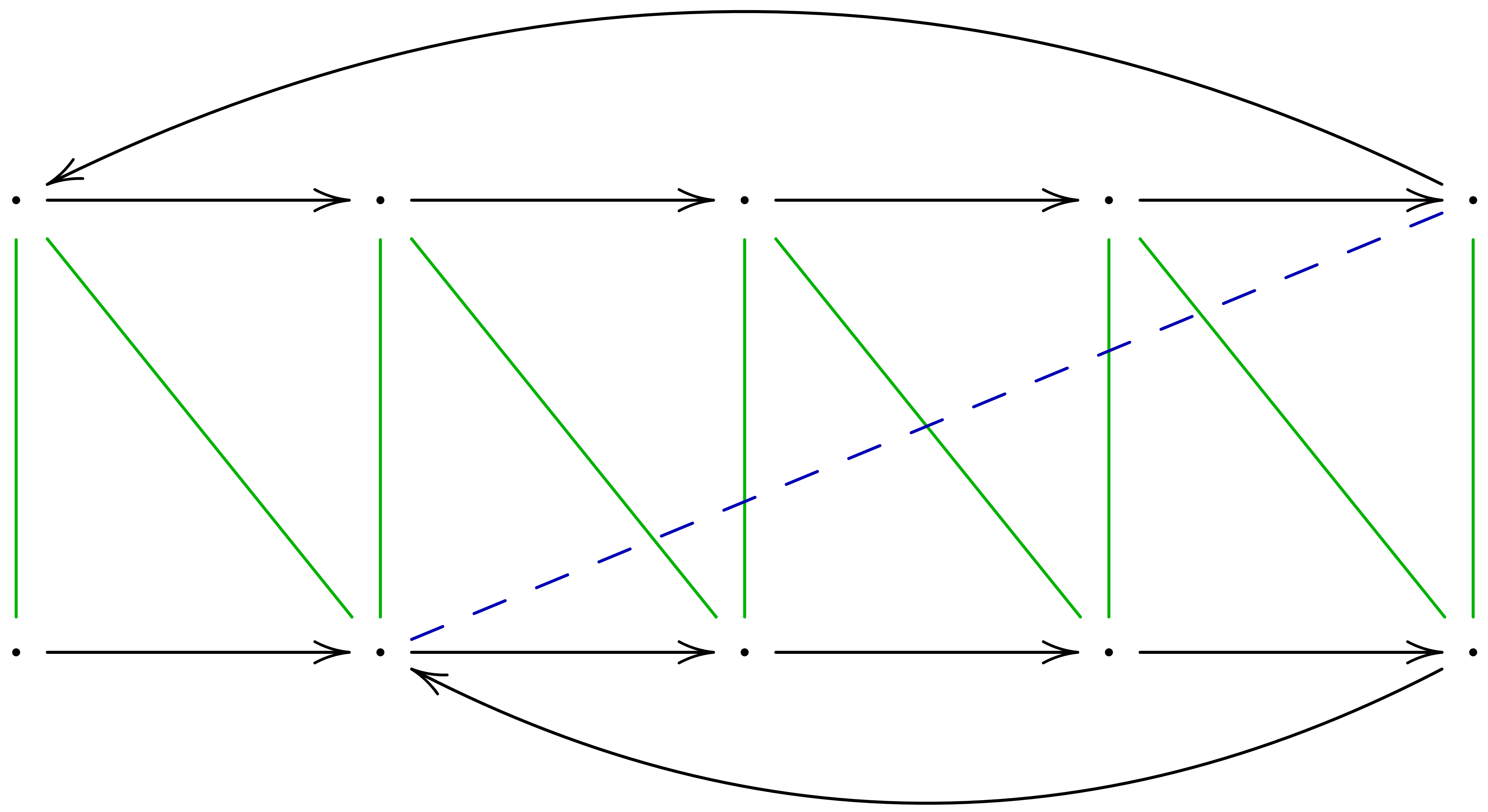


[Hopcroft and Karp '71]

Complexity: almost linear

[Tarjan '75]

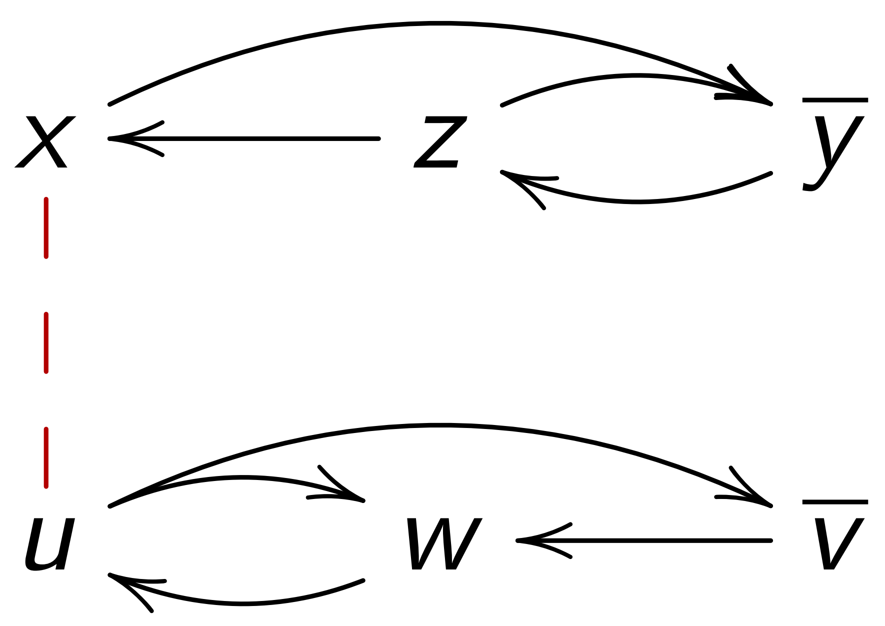
Correctness of the improvement

Correctness of HK algorithm, revisited:

- The previous relation is not a bisimulation - proof of equivalence
- But can be completed to one using equivalence - transitivity
- Hopcroft and Karp's algorithm ('71) attempts to construct a bisimulation up to equivalence

Non-Deterministic Automata

Use Hopcroft and Karp on the fly, through the powerset construction:



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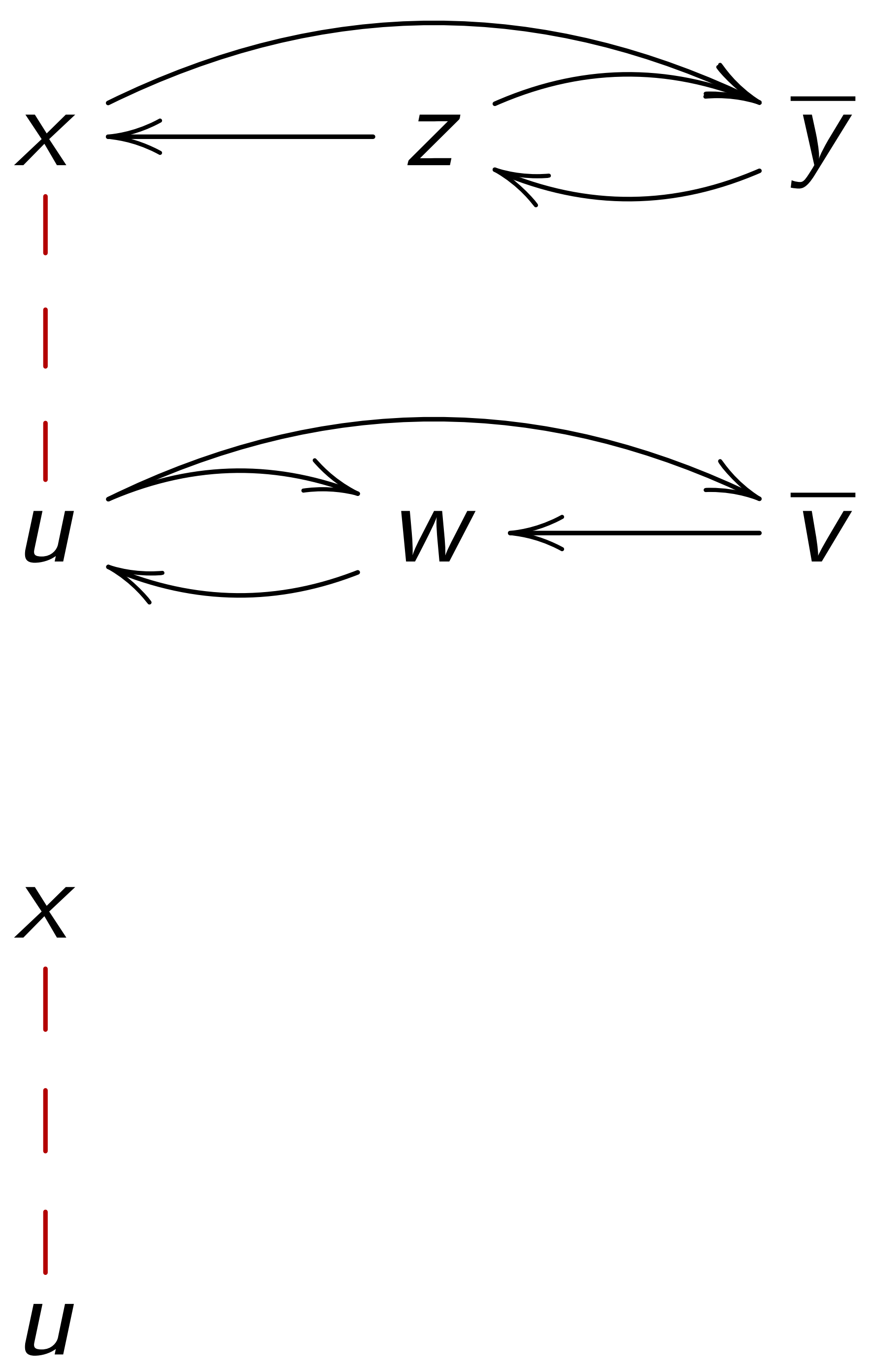

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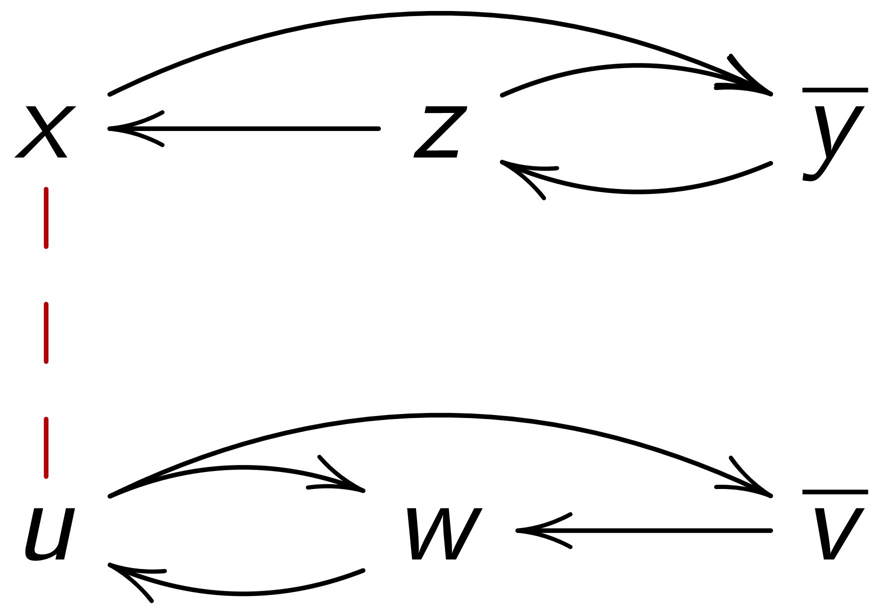

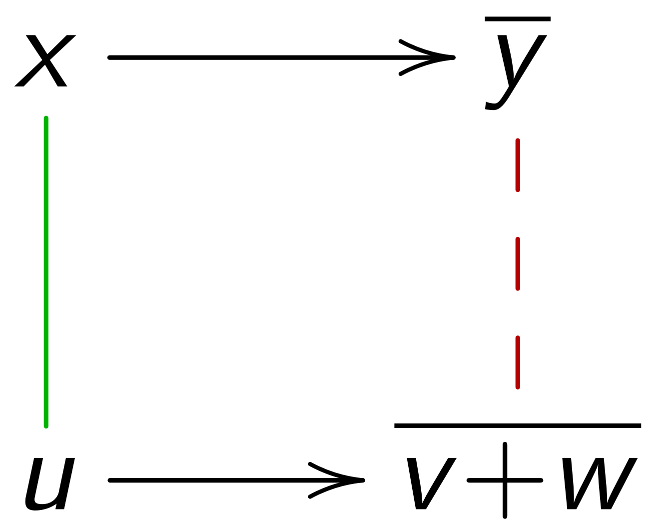

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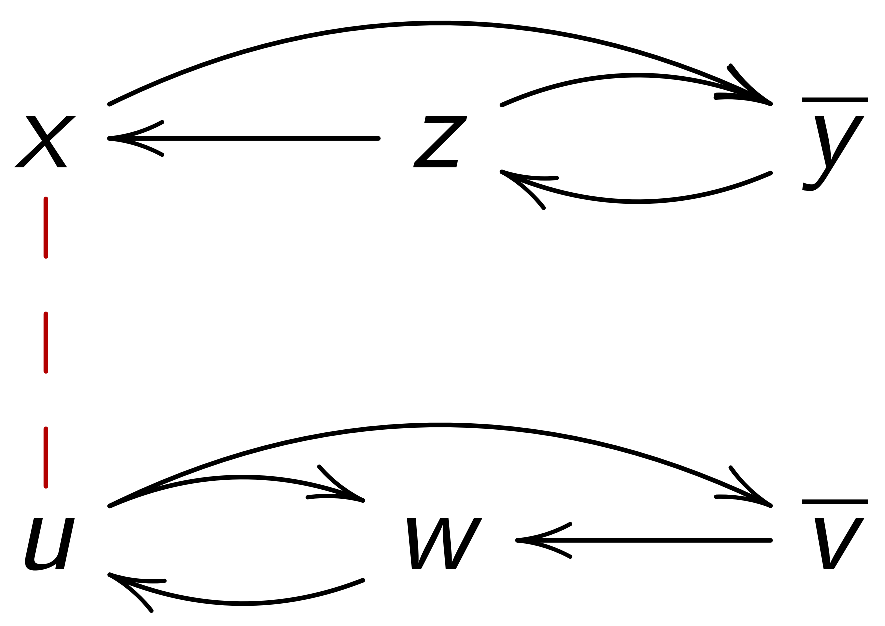

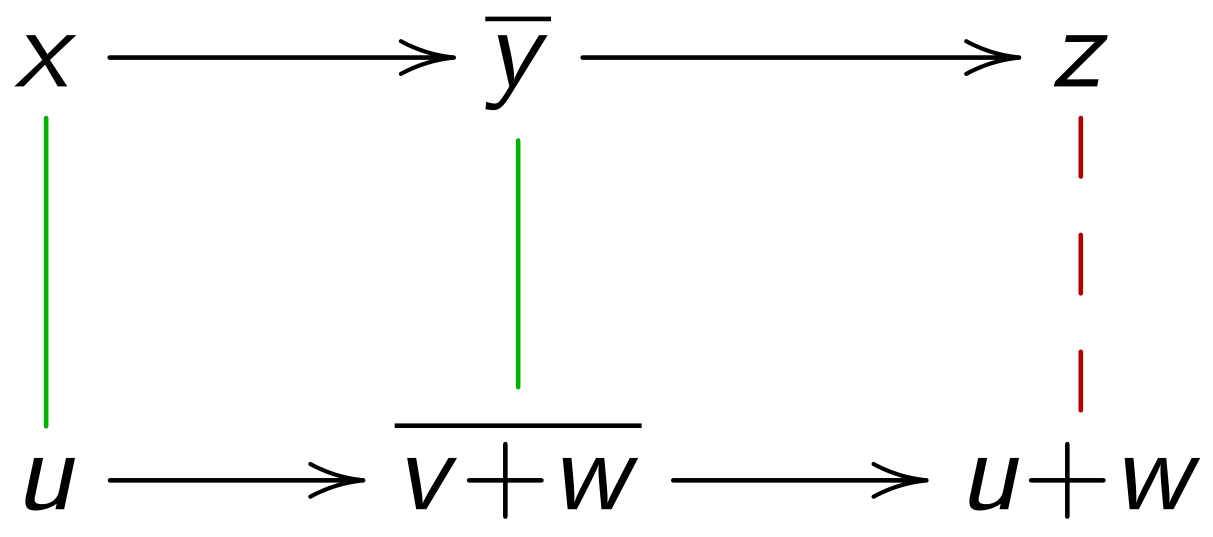

Non-Deterministic Automata

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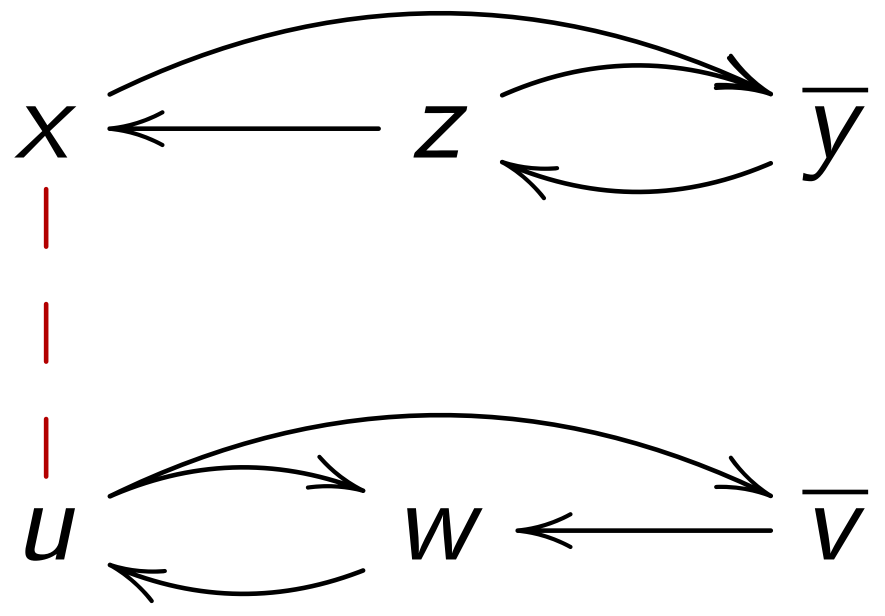

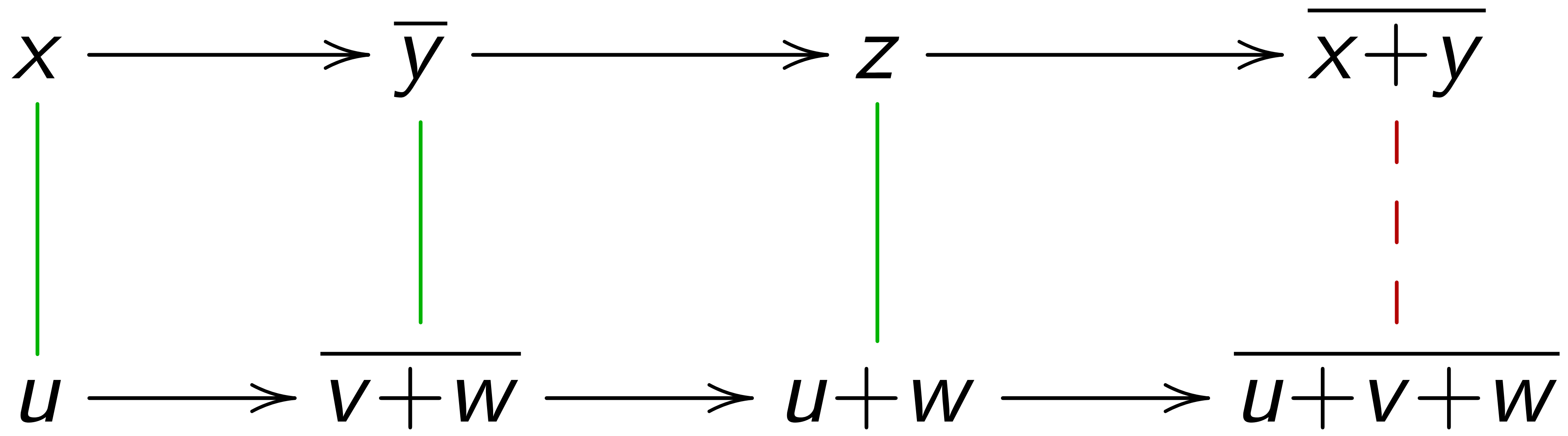

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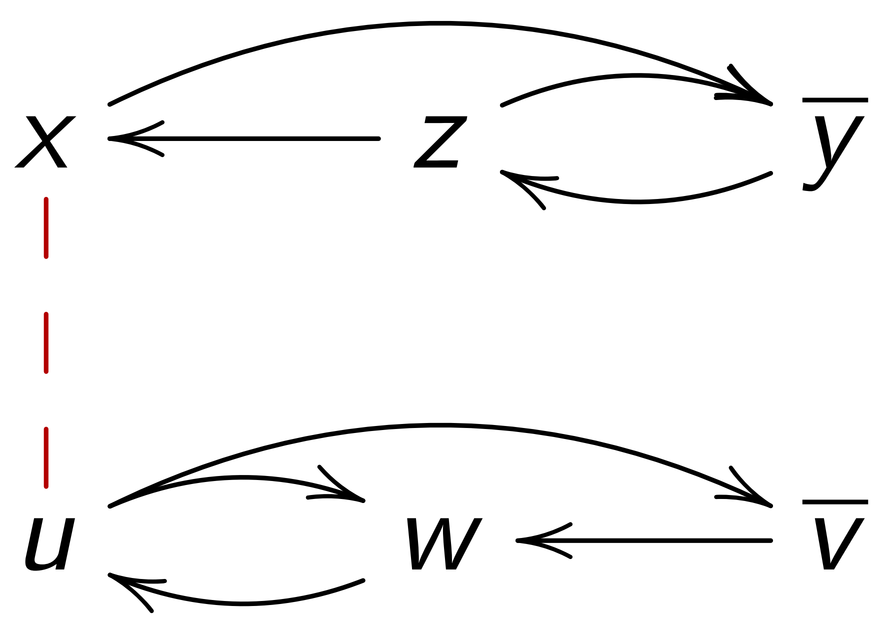

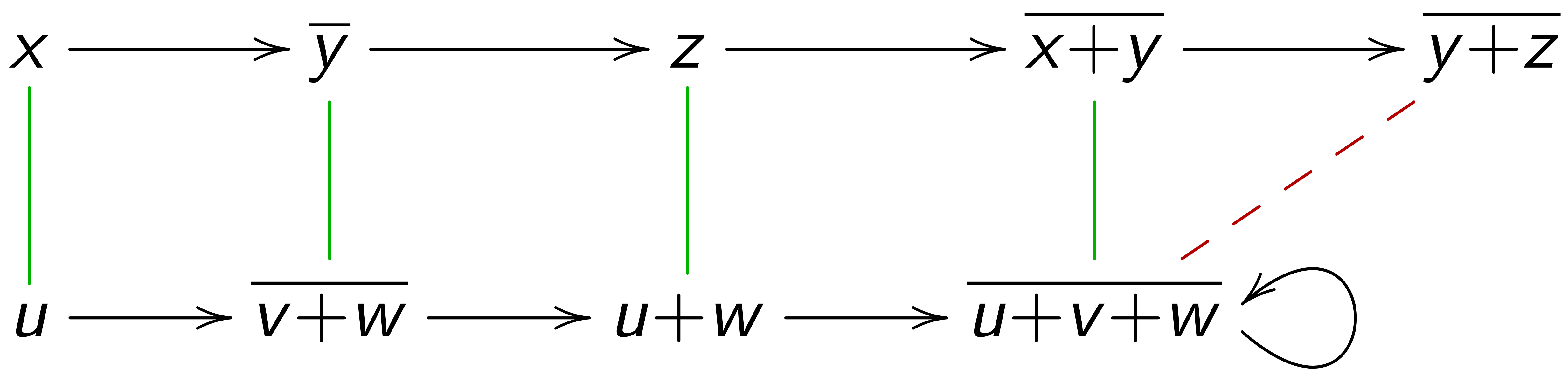

Non-Deterministic Automata

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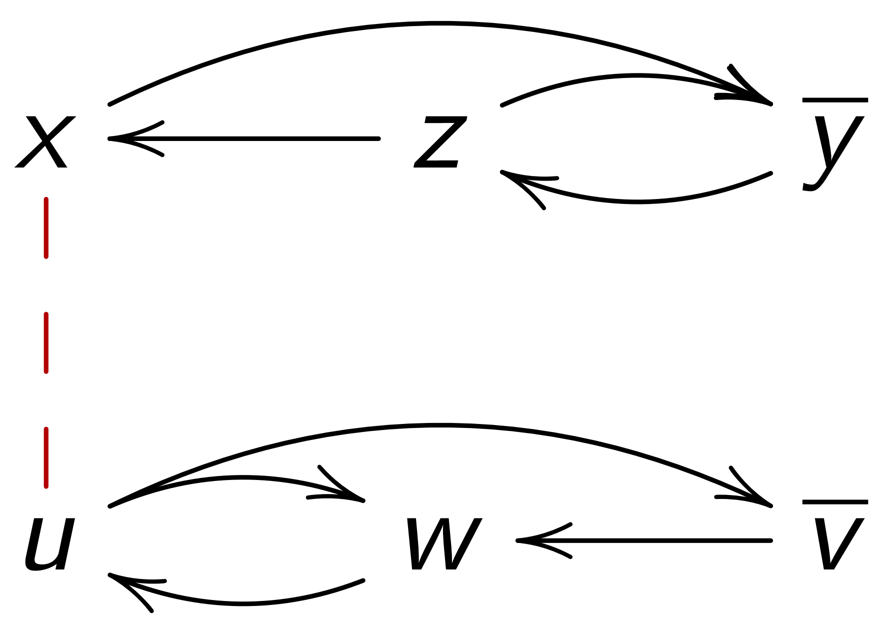

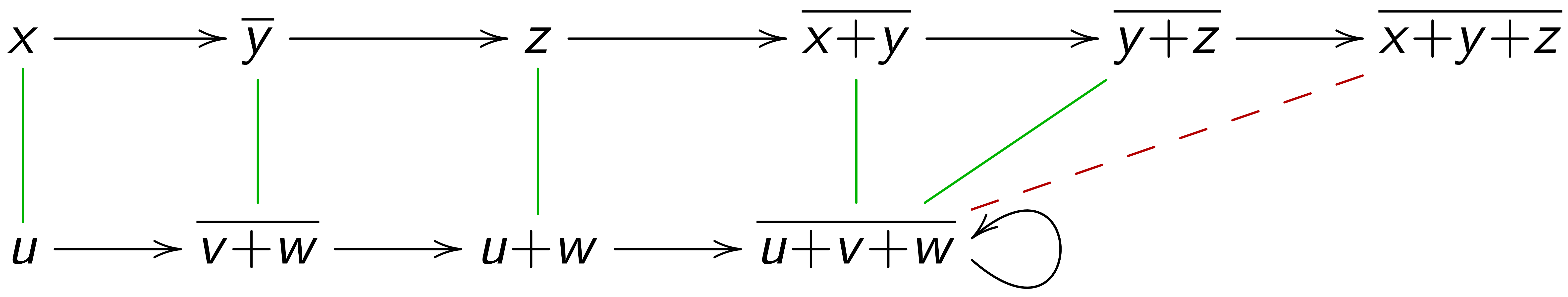

Non-Deterministic Automata

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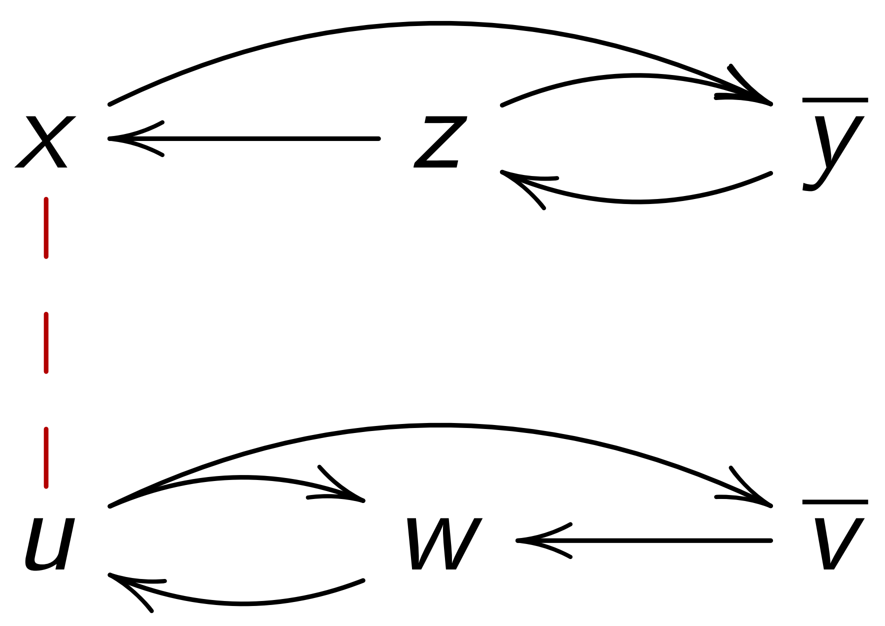

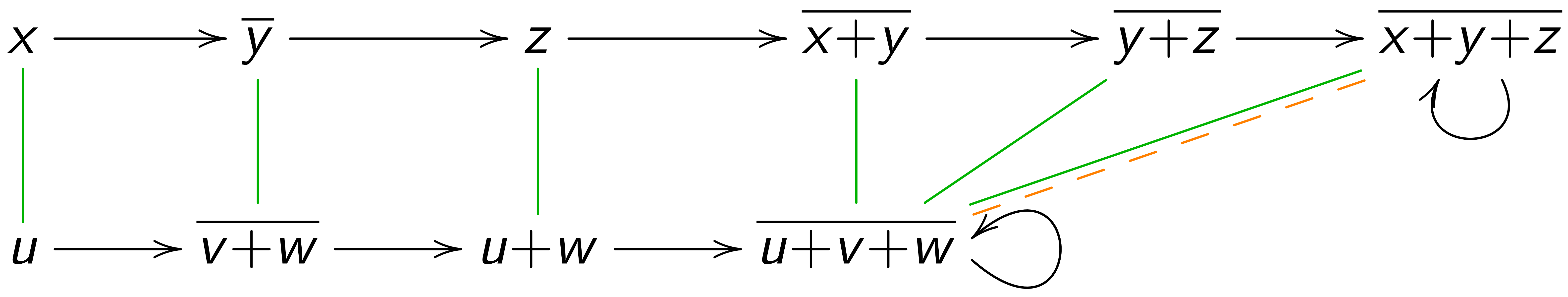

Non-Deterministic Automata

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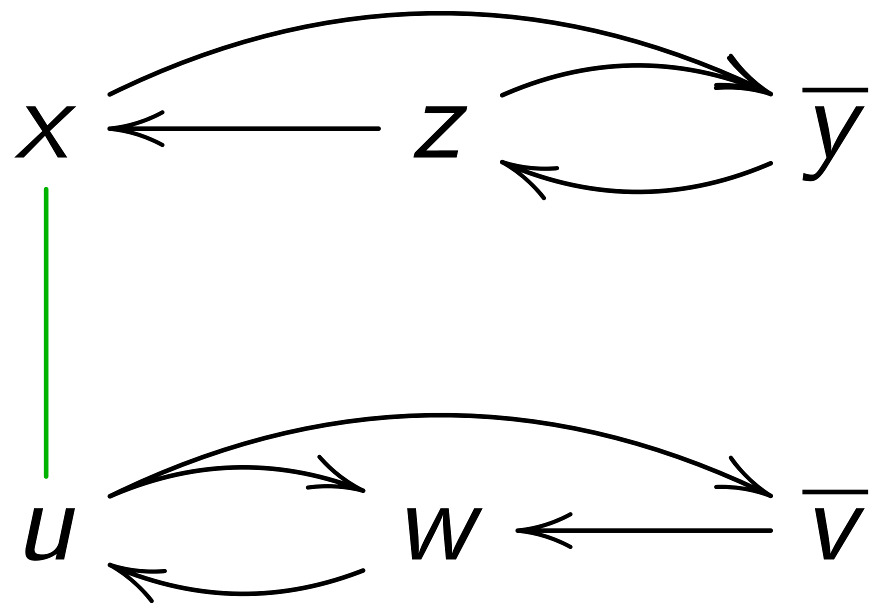
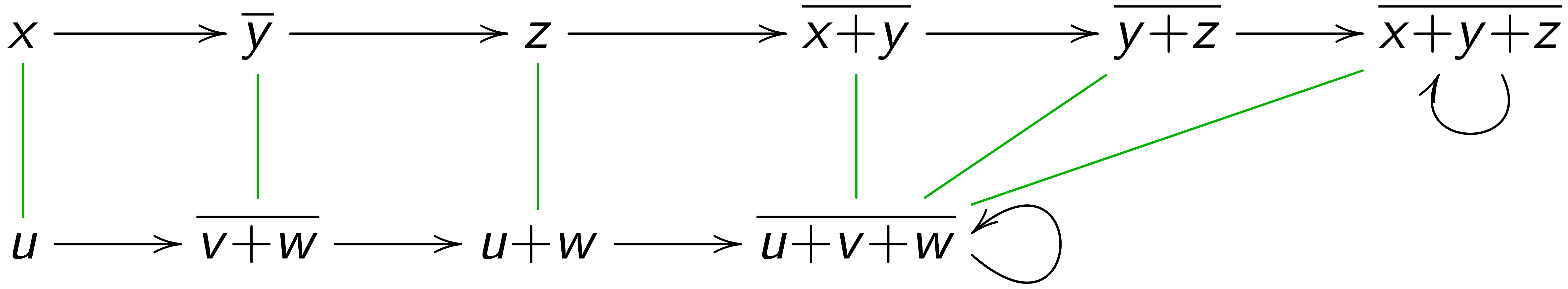
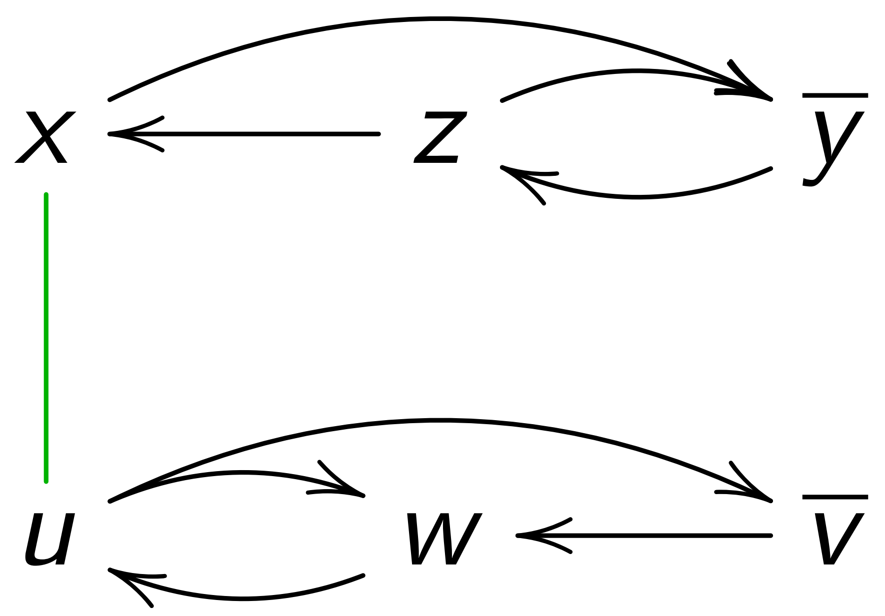
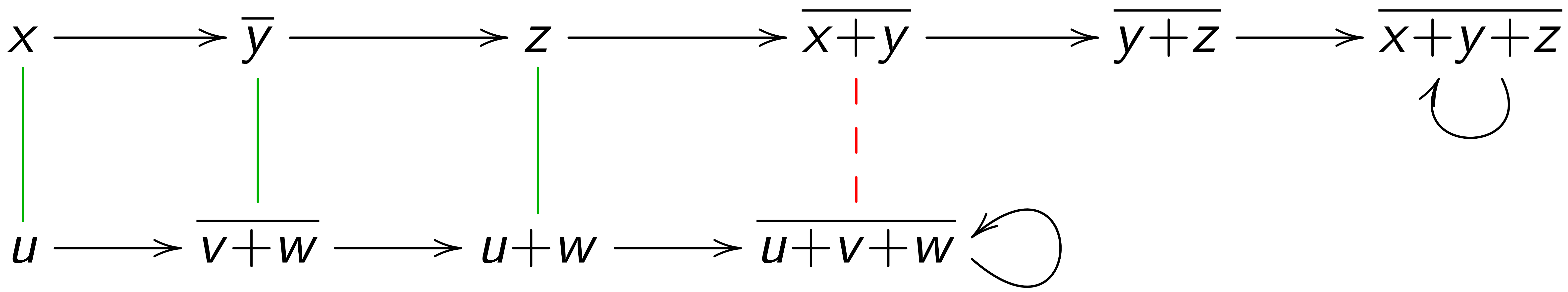




Non-Deterministic Automata

One can do better:





Non-Deterministic Automata

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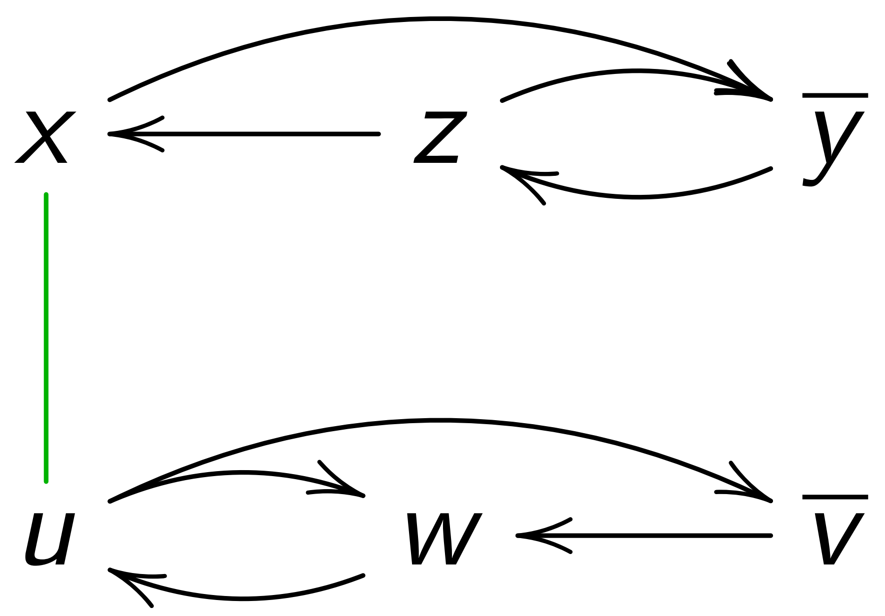
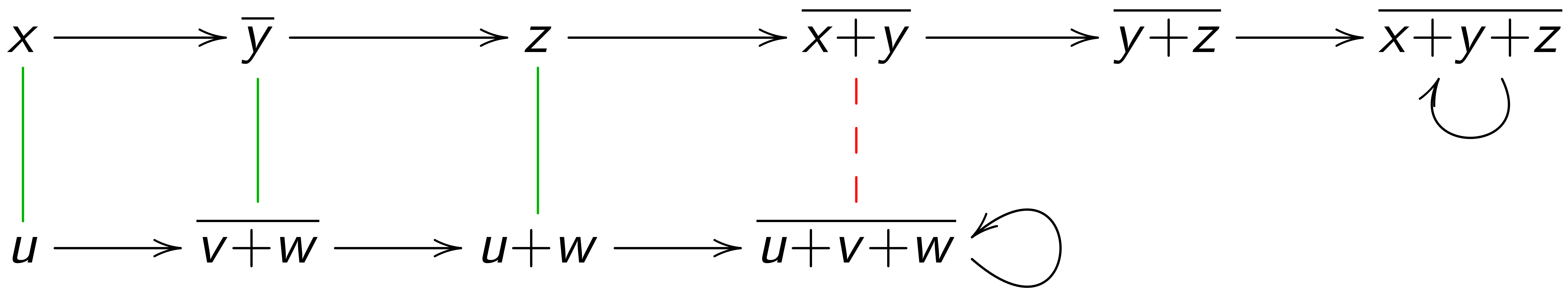


$$\begin{array}{rl} & (x,\ u) \\ + & (y,\ v+w) \\ \hline = & (x+y,\ u+v+w) \end{array}$$



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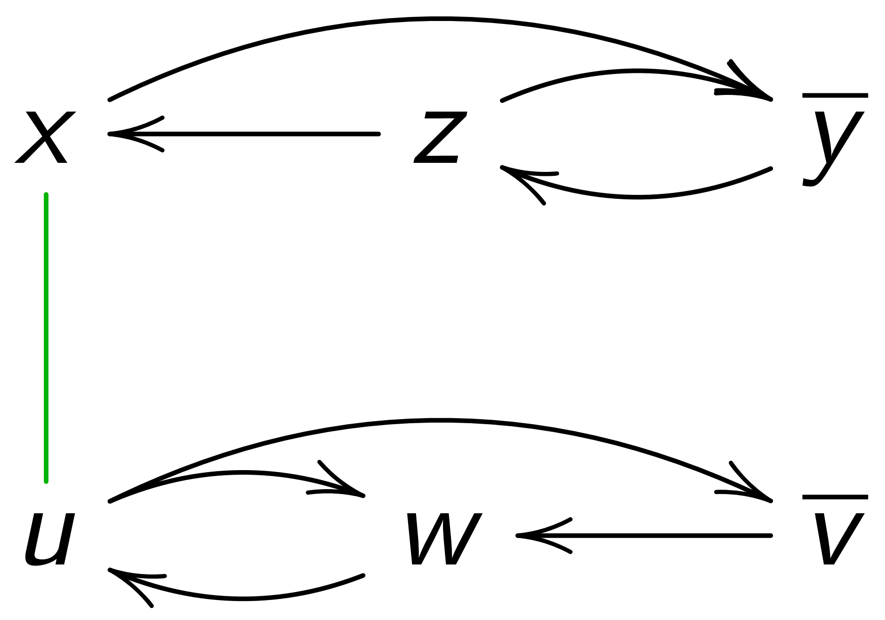
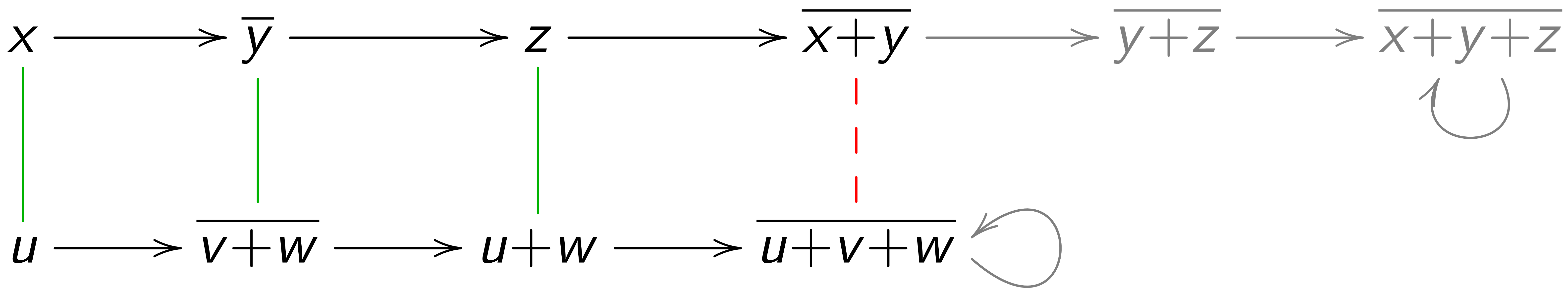


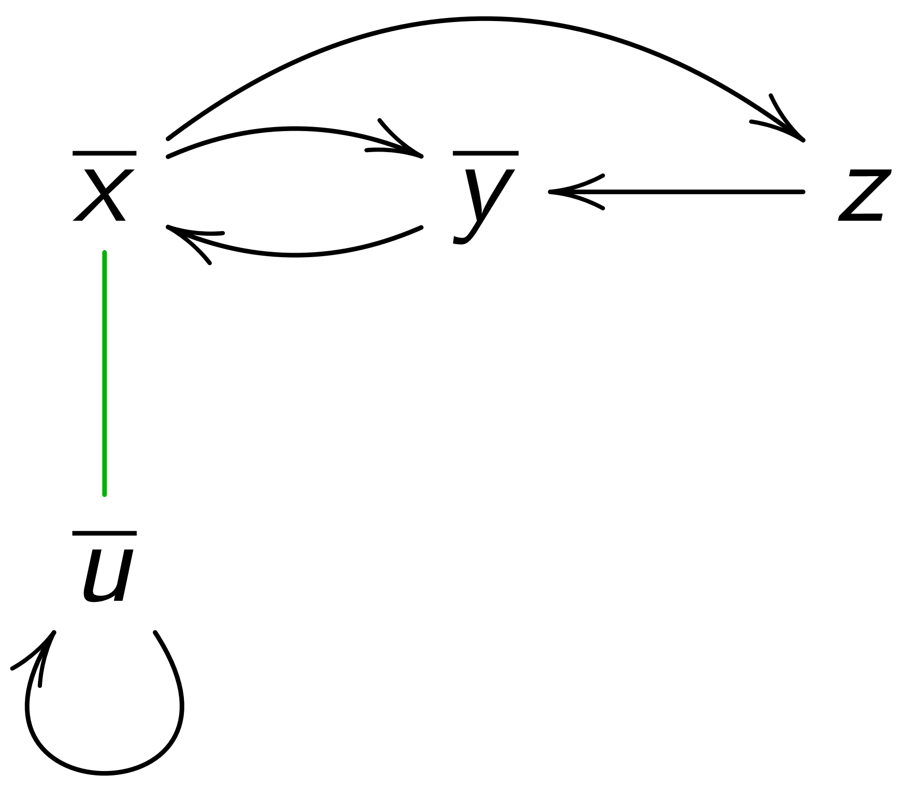
$$
\begin{array}{rl}
 & (x,\ u) \\
+ & (y,\ v+w) \\
\hline
= & (x+y,\ u+v+w)
\end{array}
$$

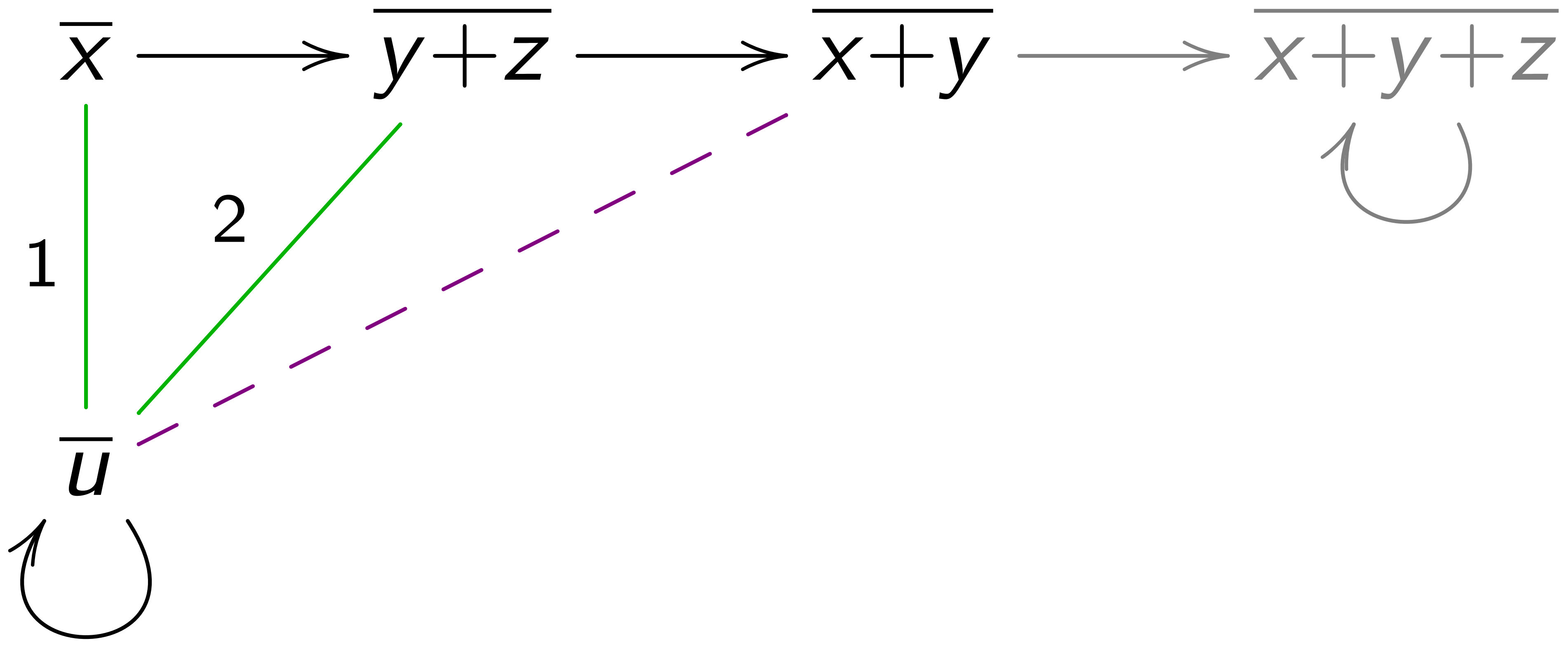


using bisimulations up to union

Non-Deterministic Automata

One can do even better:





Non-Deterministic Automata

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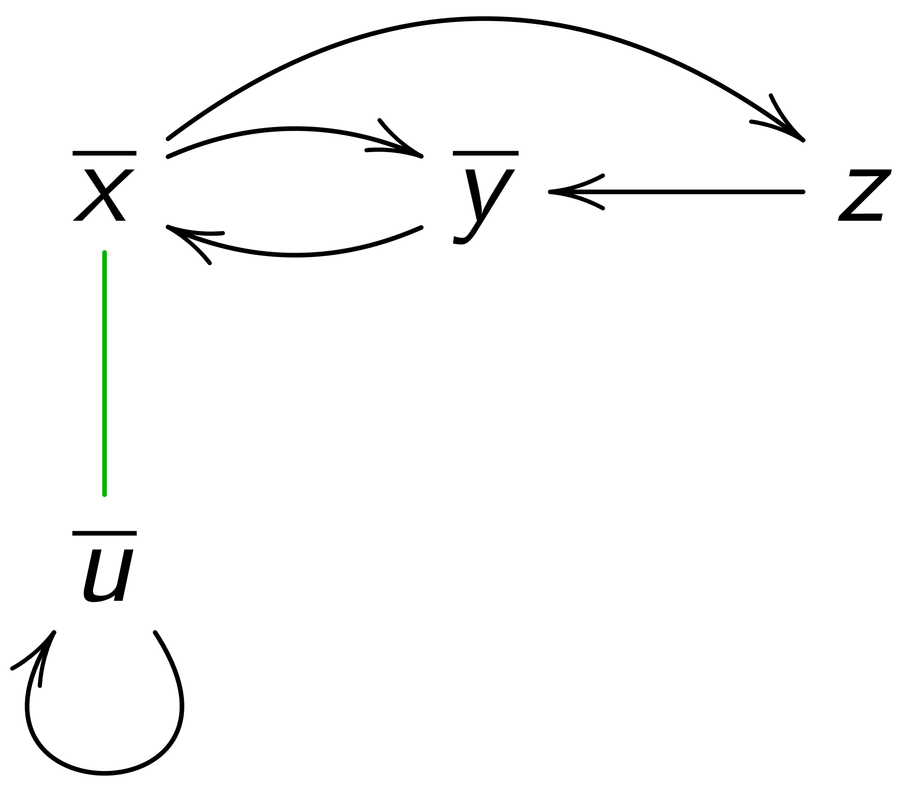
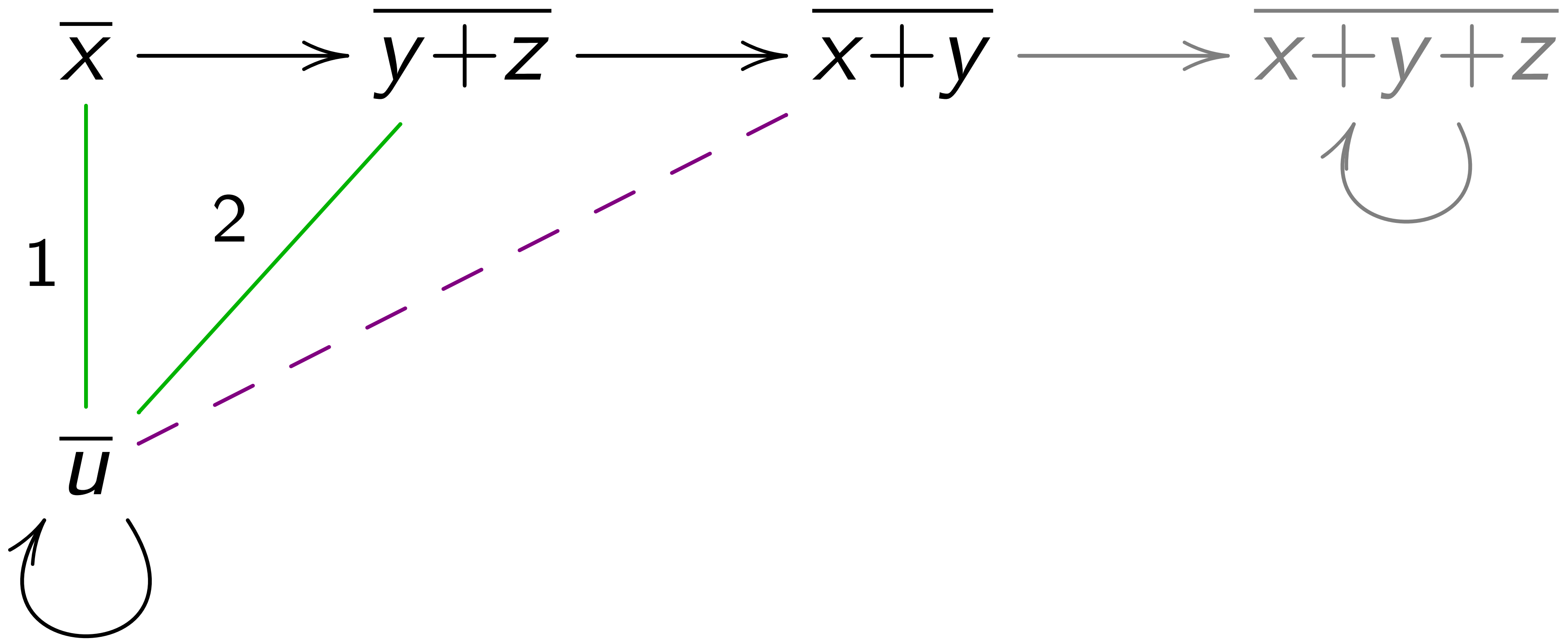


$$
\begin{aligned}
x+y &= u+y && (1)\\
&= y+z+y && (2)\\
&= y+z &&\\
&= u && (2)
\end{aligned}
$$



Non-Deterministic Automata

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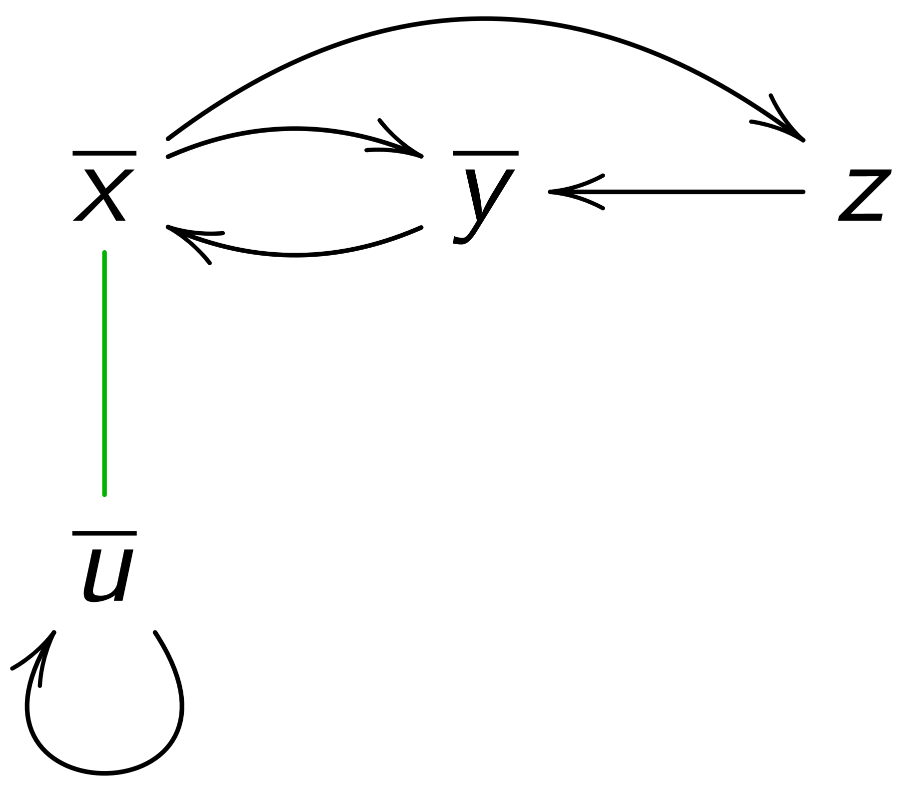

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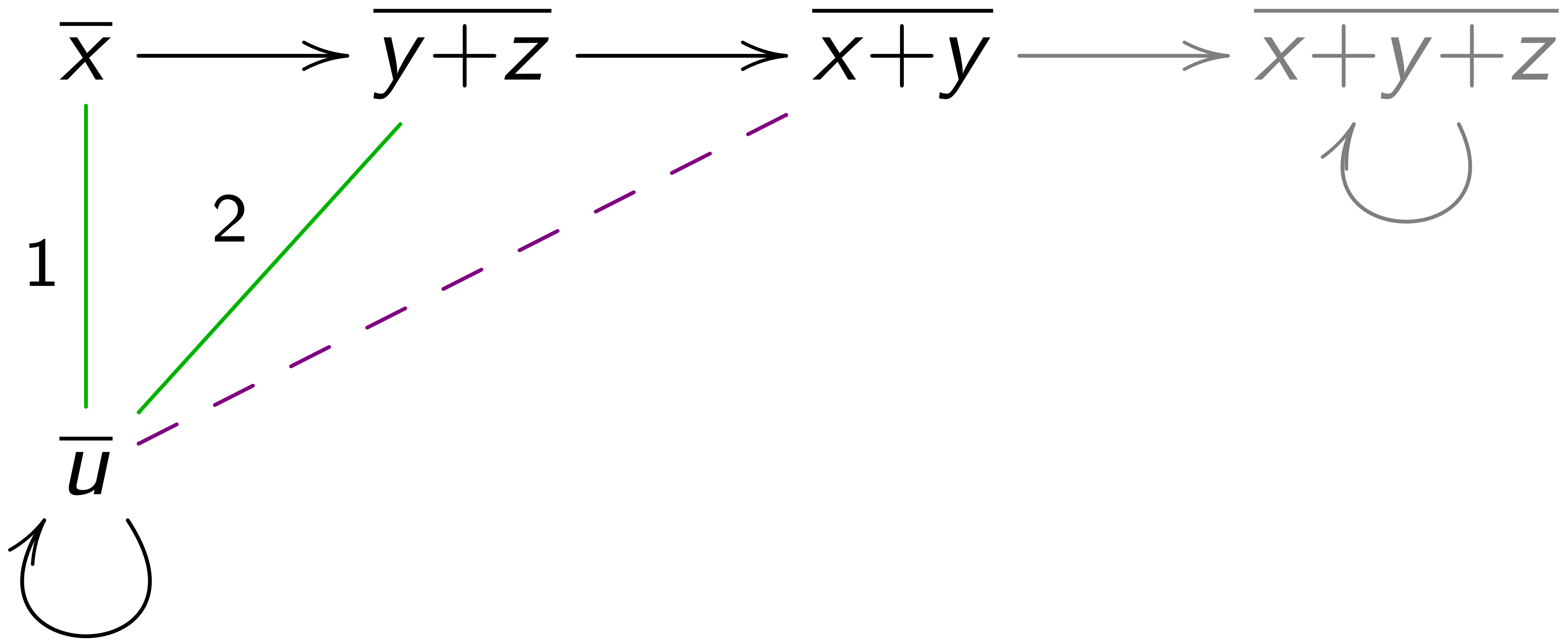

Non-Deterministic Automata

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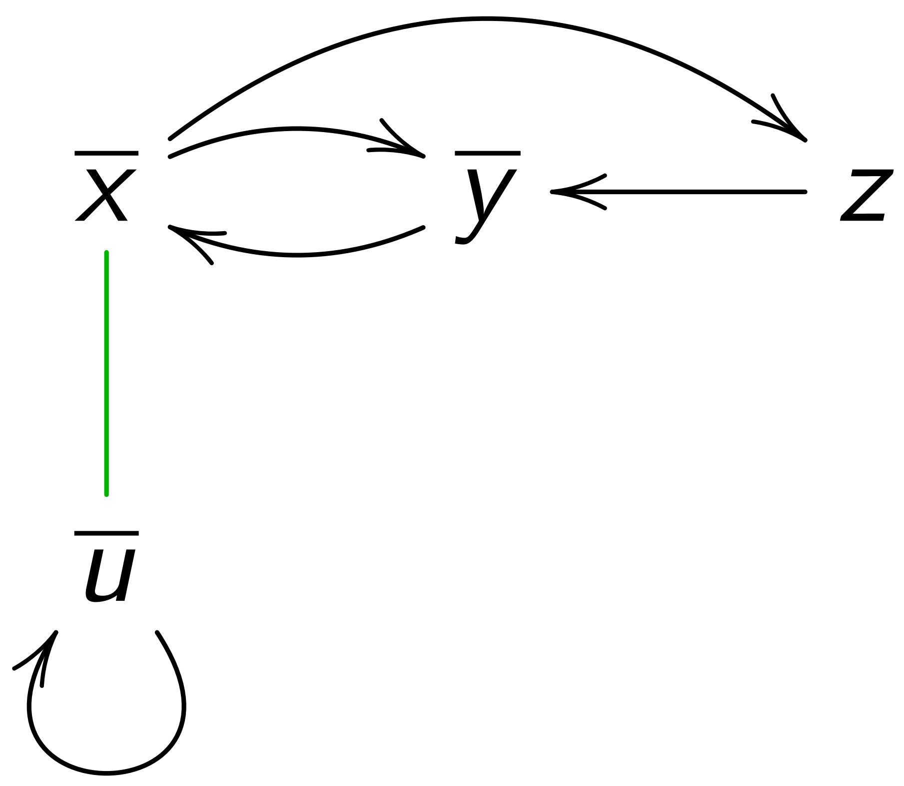

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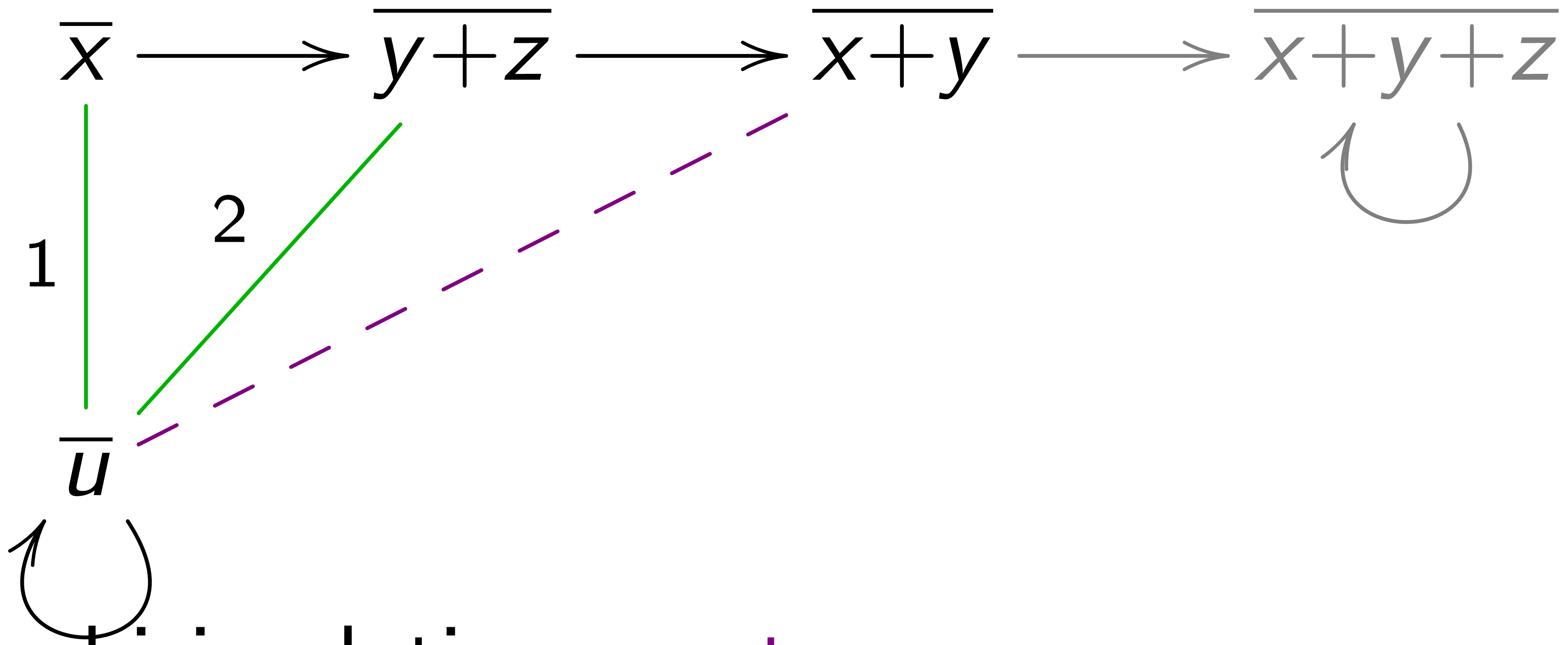

using bisimulations up to congruence

Non-Deterministic Automata

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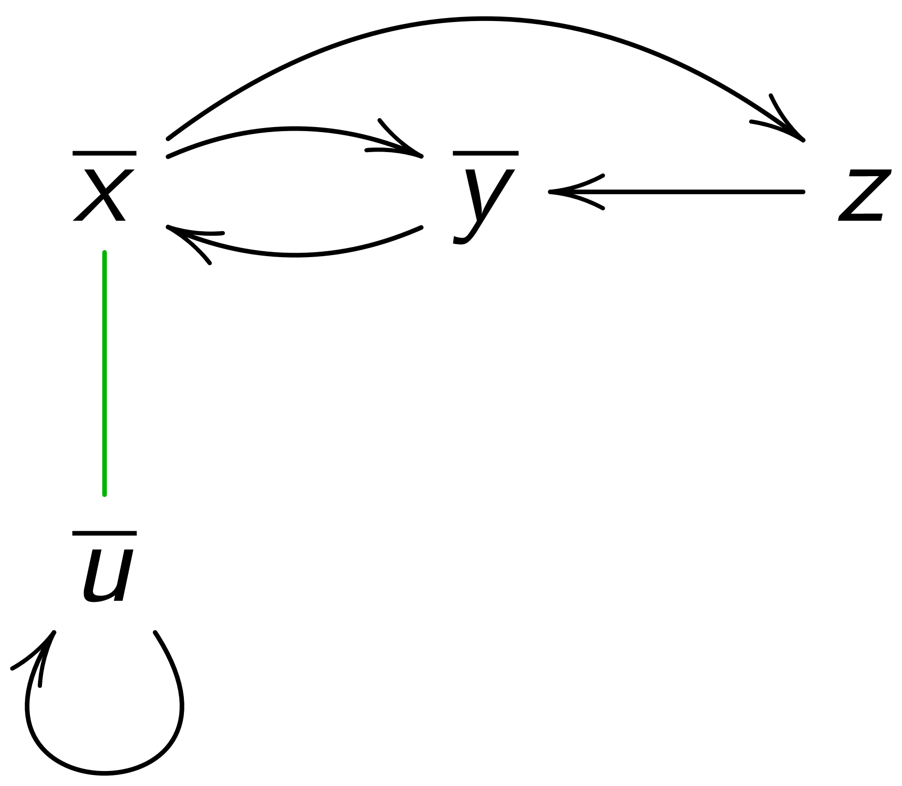

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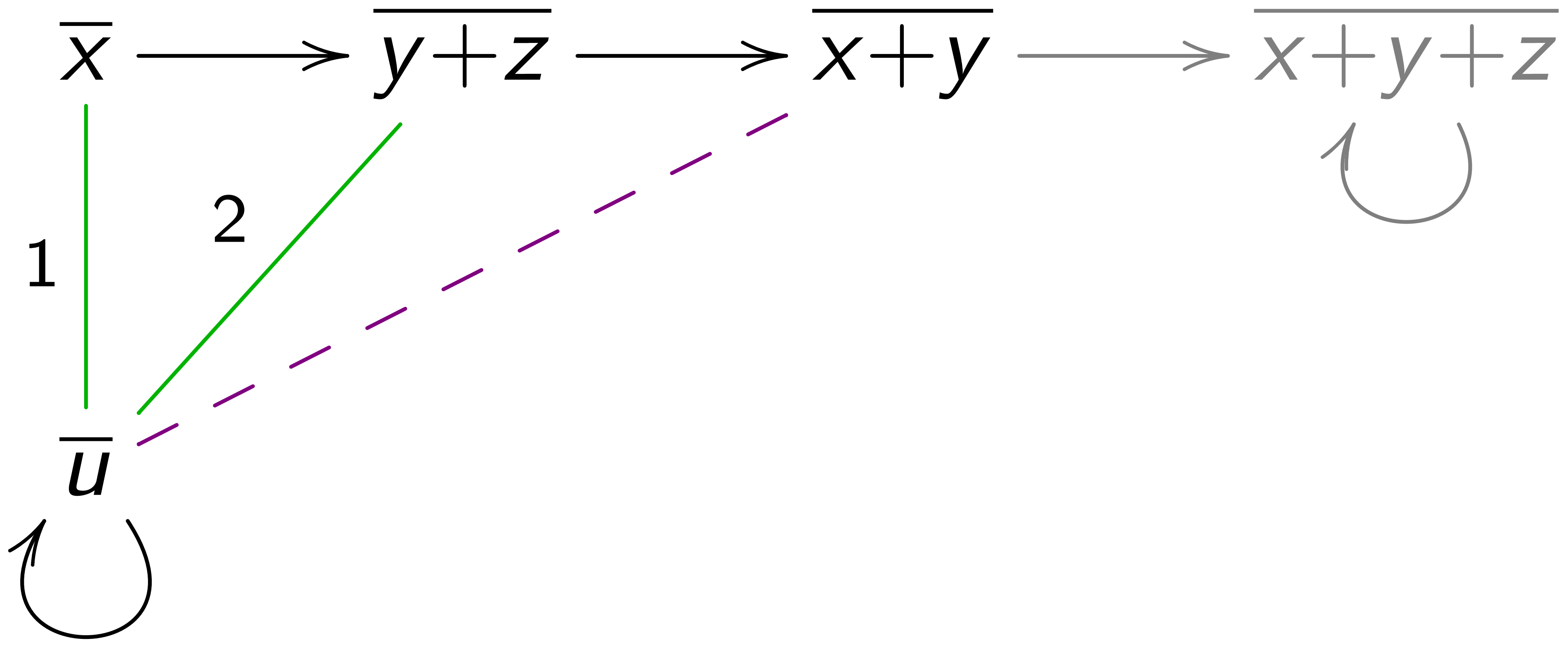

this yield to the HKC algorithm [Bonchi, Pous'13]

Intermezzo: conduction up-to with coalgebra

Next time

Lecture 3

Equivalence via axioms
Completeness