OPLSS22: Game Semantics - Lecture 3

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1 Plays and strategies

A legal play in a strategy is defined as a sequence of alternative moves where each move is tagged with pointer information. An arena A is defined by a set of moves where each move has a player or opponent polarity, and an enabling relation that describes the causal relation between certain moves (i.e. what is a valid response to a move). The set of all legal plays in arena A is denoted as L_A. A play always starts with an opponent move. For example, an arena for $\sigma \to \tau$ will have only initial moves of τ; if τ is a base type, it has a unique initial move. A strategy in such a type will always be a tree.

1.1 Legal Interactions

Legal interactions describe how to compose multiple strategies. Intuitively, a legal interaction maintains the invariant that when we remove a move from a strategy, the remaining sequence of moves is legal. For this, we look at sequences of moves that belong to arenas A, B, and C. A legal interaction (interaction for short) over these arenas is a sequence u of moves from the three arenas such that

$$u \restriction_{A,B} \in L_{A\to B} \quad u \restriction_{B,C} \in L_{B\to C} \quad u \restriction_{A,C} \in L_{A\to C}$$

Here, $u \restriction_{A,B}$ denotes the subsequence consisting only of moves in A and B (and analogously for $u \restriction_{B,C}$ and $u \restriction_{A,C}$). One should take care in maintaining the moves of A, B, and C to be all distinct, and could accomplish this by tagging moves (if needed). The set of legal interactions over A, B, and C is written as $int(A, B, C)$.

1.2 Composition of strategies

Let A, B, C be three arenas, and let σ and τ be strategies of $A \to B$ and $B \to C$. The composition of strategies $\tau \circ \sigma$ would therefore be a strategy in the space $A \to C$, and we define it as follows. u and v are interaction sequences where $u \restriction_{A,B}$ and $u \restriction_{B,C}$ are legal plays. In particular, we say that u is a witness of v.

$$\tau \circ \sigma = \{v \mid \exists u \in \mathit{int}(A,B,C)\ v = u \restriction_{A,C},\ u \restriction_{A,B} \in \sigma,\ v \restriction_{B,C} \in \tau\}$$

Note that composability of strategies complements strong evaluation as introduced in last lecture (see example Böhm tree for partial evaluation).

1.3 Associativity of composition of strategies

If there are sequences $u \in \mathit{int}(A,C,D)$ and $v \in \mathit{int}(A,B,C)$, and if we know that restricting u to (A,C) is the same thing as restricting v to (A,C) then there exists a unique interaction sequence relative to the 4 arenas (A,B,C,D) s.t. when restricting the sequence to (A,B,C) we get v and when restricting to (A,C,D) we get u. This result leads us to prove associativity. That is, given the strategies $\sigma : A \to B, \tau : B \to C, \upsilon : C \to D$, we have:

$$\upsilon \circ (\tau \circ \sigma) = (\upsilon \circ \tau) \circ \sigma$$

1.4 Identity strategy

Let A be an arena and consider words u over the alphabet $A_1 \cup A_2$. We define the identity strategy as follows:

$$\mathit{id}' = \{u \in L_{A \to A} \mid v \restriction_1 = v \restriction_2 \text{ for all even prefixes } v \text{ of } u\}$$

where $v \restriction_i$ is the result of removing the i tags from $v \restriction_{A_i}$ (for $i = 0, 1$). We can also define the identity strategy inductively

$$\epsilon \in \mathit{id}'' \qquad \frac{v \in \mathit{id}'' \quad a \text{ is an O move}}{v a_2 a_1 \in \mathit{id}''} \qquad \frac{v \in \mathit{id}'' \quad a \text{ is a P move}}{v a_1 a_2 \in \mathit{id}''}$$

Both definitions of id (id' and id'') are equivalent.

1.5 Views

The view of a play s is defined inductively over s as follows:

1)	$\ulcorner \epsilon \urcorner = \epsilon$	
2)	$\ulcorner sn \urcorner = \ulcorner s \urcorner n$	(if n is a P move)
3)	$\ulcorner sm \urcorner = m$	(if m is an O move and initial)
4)	$\ulcorner sns'm \urcorner = \ulcorner sn \urcorner m$	(if m us an O move, but not initial)

Figure 1 shows a play (top) with its view (bottom). Moves $q_{11} 3_{11}$ are removed from the view by rule 4.

1.6 Properties

Determinism A strategy σ is called deterministic if for every opponent move there is at most one valid player move:

$$smn_1, smn_2 \in \sigma \implies n_1 = n_2$$

where m is an O move while n_1 and n_2 are P moves.

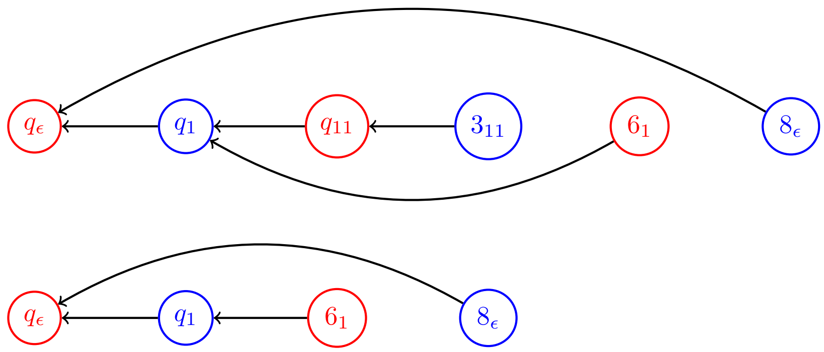

Figure 1: A play (top) with its view (bottom)

Innocence A strategy σ is said to be *innocent* if for every play s the following holds

$$s \in \sigma \iff \ulcorner s \urcorner \in \sigma$$

1.7 Fat vs meager strategies

In the canonical formulation of HO semantics, the denotation of a PCF Böhm tree (as denoted in a prior lecture, a PCF term) is the set of all plays whose view is in the transcription of the tree as a strategy. Such a set of general plays is called the *fat* representation. We have been using the *meager* definition, which only includes views. For the *innocent* strategies, the fat and meager representations are in bijection.

The "denotational" (i.e. without reference to an abstract machine) definition of composition just presented above used fat representations. For innocent strategies, the set of views of this composition is the same as the set of views we would have obtained by composing (the meager representations of) the two strategies using the abstract machine described in the preceding lecture.

2 Categorical interpretation

We obtain a category with

- Arenas as objects
- Strategies of $A \to B$ as morphisms from A to B
- Composition of strategies as composition
- The identity strategy id as identity