Rewriting and termination in lambda calculus

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Roadmap

- Rewriting basics
- Lambda calculus
- Confluence: Church-Rosser property, local confluence
- Normalisation: strong normalisation, normalisation
- Strategies: leftmost-outermost strategy, perpetual strategies
- Simple types in lambda calculus: strong normalization
- Intersection types in lambda calculus: complete characterisations of normalisations

Rewriting basics

Higher-order rewriting systems - Lambda calculus

Rewriting

Methods of replacing

- subexpressions of a formula (object)
- with other expressions

Examples: algebra, arithmetics, logic, geometry, linguistics, physics

- Term rewriting
- Higher-order term rewriting
- Graph rewriting
- String rewriting, semi-Thue systems
- Trace rewriting, concurrent computation and process calculi
- ...

Algebra - commutative group theory

$(a \star b) \star c \rightarrow a \star (b \star c)$	associativity
$a \star e \rightarrow a$	neutral element
$a \star a^{-1} \rightarrow e$	inverse element
$a \star b \rightarrow b \star a$	commutativity

Logic - propositional logic

$A \wedge B \rightarrow A$	$A \wedge B \rightarrow B$	conjunction
$A \rightarrow A \vee B$	$B \rightarrow A \vee B$	disjunction
$A \rightarrow \neg\neg A$	$\neg\neg A \rightarrow A$	negation

Abstract rewrite

$$f(x, x) \rightarrow g(x)$$
$$f(x, g(x)) \rightarrow b$$
$$h(a, x) \rightarrow f(h(x, a), h(x, x))$$

Abstract rewrite systems - ARS

$(A, \longrightarrow)$

- A is a set
- $\longrightarrow$ is a binary relation on A
  - $t \longrightarrow s$ is a step
- $\twoheadrightarrow$ is a transitive-reflexive closure
  - $t \twoheadrightarrow s$ is a sequence $t \longrightarrow s_1 \longrightarrow s_2 \ldots \longrightarrow s_n \equiv s$
- $\longrightarrow_1 \cdot \longrightarrow_2$ is a composition $t \longrightarrow_1 u \longrightarrow_2 s$
- $\longrightarrow$ is deterministic
  - if for each $t \in A$ there is at most one $s \in A$ such that $t \longrightarrow s$

ARS main properties

We focus on:

- confluence
- normalisation
- termination

Confluence

$(A, \longrightarrow)$ is confluent if $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$

$$\begin{tikzcd}
 & M \arrow[dl, ->>] \arrow[dr, ->>] & \\
P \arrow[dr, ->>] & & Q \arrow[dl, ->>] \\
 & N &
\end{tikzcd}$$

$(A, \longrightarrow)$ is locally confluent if $\longleftarrow \cdot \longrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr] & \\
P \arrow[dr, ->>] & & Q \arrow[dl, ->>] \\
 & N &
\end{tikzcd}$$

Normalisation and termination

- $n \in A$ is a normal form if there is no s such that $n \longrightarrow s$
- $t \in A$ has a normal form if $t \twoheadrightarrow n$ and n is a normal form, we say that n is a NF of t

- $(A, \longrightarrow)$ is weakly normalising (normalising) if every $t \in A$ has a normal form
  - at least one (rewrite) sequence from t is finite
  - **How to find?**

- $(A, \longrightarrow)$ is strongly normalising (terminating) if for every $t \in A$
  - all (rewrite) sequences from t are finite
  - **How to prove?**

- $(A, \longrightarrow)$ is not terminating (terminating) if for some $t \in A$
  - at least one (rewrite) sequence from t is infinite
  - **How to find?**

ARS more properties

Theorem

Confluence implies uniqueness of normal form

Proof.

Suppose t has two normal forms n_1 and n_2. Then $n_1 \twoheadleftarrow t \twoheadrightarrow n_2$. Then by confluence there is an s such that $n_1 \twoheadrightarrow s \twoheadleftarrow n_2$, which is not possible since n_1 and n_2 are normal forms.

□

Theorem (Newman's Lemma)

A terminating ARS is confluent if and only if it is locally confluent

Proof.

Based on well-founded induction □

Example

Rewrite system

$$\begin{aligned} f(x,x) &\to g(x) \\ f(x,g(x)) &\to b \\ h(a,x) &\to f(h(x,a),h(x,x)) \end{aligned}$$

- Normalizing: $h(a,a)$ has two NF b and $g(b)$
- Not terminating: $h(a,a)$ has an infinite derivation
- Not confluent: $h(a,a)$ has two NF, then it is not confluent

Reduction strategies

Reduction strategy: A restriction to a subreduction

$$\longrightarrow_e \subseteq \longrightarrow$$

is a way to control that in a term there are different possible choices of reduction

Example. CBN, CBV, perpetual, leftmost *(later)*

Normalisation strategy: A reduction strategy which reaches the normal form if it exists

Example. Leftmost *(later)*

Term rewriting

A term rewriting system (TRS) is a rewriting system (ARS) whose objects are terms, which are expressions with nested sub-expressions.

Example Logic - propositional, above, is a term rewriting system

- the terms are composed of binary operators ($\vee$) and ($\wedge$) and the unary operator ($\neg$)
- the rules contain variables, which represent any possible term (though a single variable always represents the same term throughout a single rule)

Higher-order term rewriting systems (HOR) are a generalization of first-order term rewriting systems to systems allowing

- higher order functions
- bound variables

Higher-order rewriting systems - models of computation 1930s

- Alonzo Church: lambda calculus
  - theory of functions - formalisation of mathematics
  - successful model for computable functions
- Moses Schonfinkel (1921): combinators
- Haskell Curry: combinatory logic

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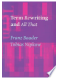
F. Baader, T. Nipkow 

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Rewriting basics

Higher-order rewriting systems - Lambda calculus

Lambda calculus: background

In mathematics,

1. expressions with free variables
2. functions

Example.

1. $x^2 + 1 < 10$	$x^2 < 9$	$x \in (-3, 3)$	free
2. $f(x) = x^2 + 1$	$f(5) = 5^2 + 1$	$f(5) = 26$	bound

Church: $\lambda x.(x^2 + 1)$ denotes a function of x

HOR feature - bound variables

Function behaviours treated formally in lambda calculus

- The name of the argument is not important
  $f(x) = x^2 + 1$ and $f(y) = y^2 + 1$
  $\lambda x.(x^2 + 1)$ and $\lambda y.(y^2 + 1)$ must be considered as equal
  α-conversion
- Function application evaluation
  $(\lambda x.(x^2 + 1))(5)$ and $5^2 + 1$ must be considered as equal
  β-conversion
- Substitution and bindings
  $(\lambda x.(\lambda y.(x^2 + 1 + y)))(y)$ should be $\lambda w.(y^2 + 1 + w)$
  but not $\lambda y.(y^2 + 1 + y)$
  Barendregt variable convention: no variable is both free and bound

HOR feature - functions of multiple arguments

In lambda calculus, functions of multiple arguments can be treated as iterated applications of single argument functions

$f(x, y, z) = x + y + z$
$f(1, 2, 3) = 6$

f becomes $(\lambda x.(\lambda y.(\lambda z.x + y + z)))$

$(\lambda x.(\lambda y.(\lambda z.x + y + z)))(1)(2)(3)$ evaluates to
$(\lambda y.(\lambda z.1 + y + z))(2)(3)$ evaluates to
$(\lambda z.1 + 2 + z)))(3)$ evaluates to
6

currying

Syntax

$V = \{x, y, z, x_1, ...\}$ a countable set of variables
$C = \{a, b, c, a_1, ...\}$ a countable set of constants

Definition

The set Λ of (lambda) terms is inductively defined

- $V \subseteq \Lambda$ and $C \subseteq \Lambda$ atomic terms
- $M, N \in \Lambda$ then $(MN) \in \Lambda$ application
- $M \in \Lambda, x \in V$ then $(\lambda x.M) \in \Lambda$ abstraction

Conventions for minimizing the number of the parentheses:

- $M_1 M_2 M_3$ stands for $((M_1 M_2) M_3)$ application associates to left
- $\lambda x.y.M$ stands for $(\lambda x.(\lambda y.(M))$ abstraction associates to right
- $\lambda x.M_1 M_2 \equiv \lambda x.(M_1 M_2)$; application has priority over abstraction

Pure lambda calculus, if $C = \emptyset$

$$M ::= x \mid MM \mid \lambda x.M$$

Running example

$xyzx$	
$\lambda x.zx$	
$\mathbf{I} = \lambda x.x$	combinator **I**
$\mathbf{K} = \lambda xy.x$	combinator **K**
$\mathbf{S} = \lambda xyz.xz(yz)$	combinator **S**
$\Delta = \lambda x.xx$	selfapplication
$\mathbf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$	fixed point combinator
$\Omega = \Delta\Delta = (\lambda x.xx)(\lambda x.xx)$	higher-order function

Free and bound variables

Definition

(i) The set $FV(M)$ of free variables of M is defined inductively:

- $FV(x) = \{x\}$
- $FV(MN) = FV(M) \cup FV(N)$
- $FV(\lambda x.M) = FV(M) \setminus \{x\}$

(ii) A variable in M is bound if it is not free

- x is bound in M if it appears in a subterm of the form $\lambda x.N$

(ii) M is a closed λ-term (or *combinator*) if $FV(M) = \emptyset$
Λ^o denotes the set of closed λ-terms.

Example

- In $\lambda x.zx$, variable z is free.
- Term $\lambda xy.xxy$ is closed.

Running example: 1. free variables

M	$Fv(M)$
$xyzx$	$\{x, y, z\}$
$\lambda x.zx$	$\{z\}$
$\mathbf{I} = \lambda x.x$	$\emptyset$
$\mathbf{K} = \lambda xy.x$	$\emptyset$
$\mathbf{S} = \lambda xyz.xz(yz)$	$\emptyset$
$\Delta = \lambda x.xx$	$\emptyset$
$\mathbf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$	$\emptyset$
$\Omega = \Delta\Delta = (\lambda x.xx)(\lambda x.xx)$	$\emptyset$

Substituion

Implicit substitution, meta notion (not in the language)

Definition

$N[x := M]$ is defined by induction on the structure of N

$$
\begin{array}{llcl}
N = x & x[x := M] & = & M \\
N = y & y[x := M] & = & y \\
N = N_1 N_2 & (N_1 N_2)[x := M] & = & N_1[x := M]\ N_2[x := M] \\
N = \lambda x.N & (\lambda x.N)[x := M] & = & \lambda x.N \\
N = \lambda y.N & (\lambda y.N)[x := M] & = & \lambda y.(N[x := M]),\ y \notin FV(M) \\
N = \lambda y.N & (\lambda y.N)[x := M] & = & (\lambda z.N[y := z])[x := M]), \\
 & & & y \in FV(M)\ \wedge\ z \notin FV(M)
\end{array}
$$

Explicit substitution, calculi where substituion is in the language

Rewrite rules

α-reduction:

$$\lambda x.M \longrightarrow_{\alpha} \lambda y.M[x := y], \quad y \notin FV(M)$$

β-reduction:

$$(\lambda x.M)N \longrightarrow_{\beta} M[x := N]$$

η-reduction:

$$\lambda x.(Mx) \longrightarrow_{\eta} M, \quad x \notin FV(M)$$

Properties of rewrite systems: confluence, normal forms, normalisation, strong normalisation, strategies

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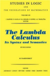
H.P. Barendregt. 

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