Rewriting and termination in lambda calculus

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Roadmap

- Rewriting basics
- Lambda calculus
- Confluence: Church-Rosser property, local confluence
- Normalisation: strong normalisation, normalisation
- Strategies: leftmost-outermost strategy, perpetual strategies
- Simple types in lambda calculus: strong normalization
- Intersection types in lambda calculus: complete characterisations of normalisations

Intersection types

There is no perfect world

In $\lambda \rightarrow$

$$\lambda x.xx : \textit{NO}$$

Intersection types

- The abstract grammar that generates the language

$$\sigma \ ::= \ \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma \mid \omega$$

Subtyping is a pre-order on types

1. $\sigma \leq \sigma$
2. $\sigma \leq \tau$, $\tau \leq \rho$, then $\sigma \leq \rho$
3. $\sigma \leq \omega$
4. $\sigma \cap \tau \leq \sigma$, $\sigma \cap \tau \leq \tau$
5. $\sigma \leq \sigma \cap \sigma$ (idempotent)
6. $\sigma \leq \sigma'$ and $\tau \leq \tau'$, then $\sigma \cap \tau \leq \sigma' \cap \tau'$
7. $\sigma' \leq \sigma$ and $\tau \leq \tau'$, then $\sigma \to \tau \leq \sigma' \to \tau'$
8. $(\sigma \to \rho) \cap (\sigma \to \tau) \leq \sigma \to \rho \cap \tau$

Type system

- Axiom

$$\frac{}{\Gamma, x : \sigma \vdash x : \sigma}\,(Ax)$$

- Rules

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\,(elim \to) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}\,(intr \to)$$

$$\frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma}\,(elim\cap) \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \tau}\,(elim\cap) \qquad \frac{\Gamma, \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}\,(intr\cap)$$

$$\frac{\Gamma, \vdash M : \sigma \quad \sigma \leq \tau}{\Gamma \vdash M : \tau}\,(\leq)$$

Intersection types

Introduced in the 1980s, to overcome the limitations of simple types

- Coppo, Dezani
- Pottinger
- Sallé

- Intersection types do not correspond to inuitionistic conjunction

$$\sigma \to \tau \to \sigma \cap \tau$$

intuitionistically provable, but not inhabited in $\lambda\cap$

$\lambda x.xx$ is typable, finally!

$$\dfrac{\dfrac{\dfrac{x : (\sigma \to \tau) \cap \sigma \vdash x : (\sigma \to \tau) \cap \sigma}{x : (\sigma \to \tau) \cap \sigma \vdash x : \sigma \to \tau}\,(elim\cap) \qquad \dfrac{x : (\sigma \to \tau) \cap \sigma \vdash x : (\sigma \to \tau) \cap \sigma}{x : (\sigma \to \tau) \cap \sigma \vdash x : \sigma}}{x : (\sigma \to \tau) \cap \sigma \vdash xx : \tau}\,(elin}{\vdash \lambda x.xx : ((\sigma \to \tau) \cap \sigma) \to \tau}$$

Intersection types - SN

Complete characterization of strong normalization

Theorem

$$M \text{ is typable} \iff M \text{ is SN}$$

Proof.

Typability $\Longrightarrow$ SN

- reducibility method
- arithmetic proof
- non-idempotent intersection types

SN $\Longrightarrow$ Typability

- typability of normal forms
- head subject expansion, perpetual strategies

- Typability is undecidable

Intersection types - normalisation properties

Complete characterization of: normalizing, head normalizing, and unsovable terms.

Theorem

- *M is normalising* $\Longleftrightarrow$ *M is typable,* $\Gamma \vdash M : \sigma$, ω *is not in* Γ *and* σ
- *M head normalising* $\Longleftrightarrow$ *M is typable,* $\Gamma \vdash M : \sigma$, $\sigma \not\sim \omega$
- *M is unsovable* $\Longleftrightarrow$ *M is typable,* $\Gamma \vdash M : \sigma$, $\sigma \sim \omega$

Proof.

Typability $\Longrightarrow$ (normalization property)

- reducibility method
- other proofs?

(normalization property) $\Longrightarrow$ Typability

- head subject expansion, leftmost strategies, perpetual strategies

Union types

- The abstract grammar that generates the language

$$\sigma \;::=\; \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma \mid \sigma \cup \sigma$$

- Rules

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M : \sigma \cup \tau}\,(intro\cup) \qquad \frac{\Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cup \tau}\,(intro\cup)$$

$$\frac{\Gamma \vdash P : \sigma \cup \tau \quad \Gamma, x : \sigma \vdash M : \rho \quad \Gamma, x : \tau \vdash M : \rho}{\Gamma \vdash M[x := P] : \rho}\,(elim\cup)$$

Theorem

$$M \text{ is typable} \iff M \text{ is SN}$$

Problems with SR!

Intersection types - more

- Curry version - implicit typing
- intersection as a proof theoretical connective (vs logical connectives)
- several proposals for the Church version - explicit typing
- bounded polymorphism
- filter models
- Böhm trees

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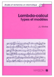
J.-L. Krivine. 

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Wrap up

- Rewrite systems
- Lambda calculus as a rewrite system
- Combintory logic as a rewrite system
- Confluence, normalisation, strong normalisation
- Standardisation, reduction strategies
- Simple types and strong normalisation
- Intersection types and normalisation properties

In lieu of conclusion

- ↬ revisit
- ↬ rethink
- ↬ rewrite