Pfenning Proof Theory Lecture 1

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1 Introduction

1.1 Why Study Proof Structure?

1. Structure of proof is fundamental to proof theory. Proof theory is the basis of all logic.

2. Constructive proof can be executed, this means that the structure of proofs is important. Two different proofs of the same proposition can have different execution if the structure is different. Therefore to understand the results from constructive proofs you must understand proof structure.

3. Proof is a means of communication among PL researchers, so understanding it is essential to communication.

1.2 Three papers

1. Gentzen 1935

2. Prawitz 1965

3. Dummett 1976

2 Defining Logical Connectives

2.1 Two approaches

1. **Verificationist (What we focus on)**: Define a logical connective by when you can conclude it, i.e. Start from the thing we want to prove, try to find what we need in order to prove it.

2. **Pragmatist**: Define a logical connective by when we can use it in a proof, i.e. Start from the thing we know is true, try to find what we can deduce from that.

2.2 Vocabulary

- **Proposition**: A statement that can be true or false. E.g. $A \wedge B$ (A *and* B)
- **Logical Connective**: The symbols that make up propositions. E.g. $\wedge, \vee$.
- **Judgment**: Something that can be proven, typically containing a proposition. E.g. that a proposition is true/false, or that a program `M` has type `A` (`M: A`).
- **Premises**: Above the bar, things that are known as true and are useful to prove the conclusion.
- **Conclusion**: Below the bar, the thing that is proven.
- **Reduction**: Transforming one set of deduction rules into another to make proofs simpler.
- **Expansion**: The inverse of reduction.
- **Derived Rule**: A deduction rule defined in terms of a set of other rules. E.g. swap.
- **Hypothetical Proof**: Proofs with non-empty premises (E.g. *if* A and B, *then* B and A).
- **Harmony**: Soundness and Completeness of introduction and elimination rules, shows that the pragmatist and verification views are in *harmony* with each other.

3 Examples of Introduction and Elimination Rules

$$\frac{A\ true \qquad B\ true}{A \wedge B\ true}\ \wedge\text{-I} \qquad \frac{A \wedge B\ true}{A\ true}\ \wedge\text{-E1} \qquad \frac{A \wedge B\ true}{B\ true}\ \wedge\text{-E2}$$

4 Verificationist Approach to Conjunction

Suppose we want to define the logical connective $\wedge$. We are taking the Verificationist approach so we begin with a rule showing when we can conclude $A \wedge B$. We define the introduction rule $\wedge$-I as follows:

$$\frac{A\ true \qquad B\ true}{A \wedge B\ true}\ \wedge\text{-I}$$

The proposition $A \wedge B$ is true if A is true and B is true. This rule is *schematic*, in that it can be used for any A and B.

4.1 Pragmatist Approach is Derivable

The pragmatist wants to know what can be derived if we know $A \wedge B$ *true*. The following are the *elimination* rules for $\wedge$:

$$\frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\ \wedge\text{-E}_1$$

$$\frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\ \wedge\text{-E}_2$$

We can see that these elimination rules are *sound* with the following proof:

$$\dfrac{\dfrac{A\ \mathit{true} \qquad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\ \wedge\text{-I}}{B\ \mathit{true}}\ \wedge\text{-E}_2$$

However if we want true harmony, we must also be able to infer the introduction rules using the elimination rules. We can see that is possible with the following proof:

$$\dfrac{\dfrac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\ \wedge\text{-E}_1 \qquad \dfrac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\ \wedge\text{-E}_2}{A \wedge B\ \mathit{true}}\ \wedge\text{-I}$$

This is good. But if we want true harmony we need soundness and completeness. Luckily doing those proofs has done that!

4.2 Soundness

Since we can derive elimination rules we have soundness. This is done using reduction.

4.3 Completeness

Did we forget something? We want to *reintroduce* the original deduction rule. As done above we can see that we need both elimination rules to do this, therefore it is complete. This is done using expansion.

Now we have seen these basic blocks for $\wedge$. Can we do this for all interesting connectives? This is what we explore in proof theory.

5 Derived Rules and Hypothetical

We have seen elimination and introduction rules, but what about rules that are neither?

Let's try a simple one, if $A \wedge B$ prove $B \wedge A$:

$$\dfrac{\dfrac{A \wedge B\ true}{B\ true}\ \wedge\text{-E}_1 \qquad \dfrac{A \wedge B\ true}{A\ true}\ \wedge\text{-E}_2}{B \wedge A\ true}\ \text{SWAP}$$

This is good, but we do have an overarching assumption that $A \wedge B$. Since we must assume $A \wedge B$ the premises are not empty. That makes this a *hypothetical* judgement.

6 Implication

Suppose, hypothetically, we want to get rid of the hypotheticals and only work in closed proofs. We can do that with implication.

For example $SWAP$ can be re-written into this closed judgement.

$$\dfrac{\dfrac{A \wedge B\ true}{B\ true}\ \wedge\text{-E}_1 \qquad \dfrac{A \wedge B\ true}{A\ true}\ \wedge\text{-E}_2}{A \wedge B\ true \to B \wedge A\ true}\ \text{SWAP} \quad .$$

Now instead of being implicit we explicitly state our premise. The implication can only be used above the judgment it implies. Is must stay in that scope. In a way closing a hypothetical by adding an implication is like changing a global variable to a scoped variable in the only place it is used.

6.1 Defining Implications

Like we did with conjunction let's formalize implication and prove that it is sound and complete. We can define the introduction rule as follows:

$$\dfrac{\begin{array}{c} A\ true : x \\ \vdots \\ B\ true \end{array}}{A \to B\ true}$$

For clarity we have named the proposition $A\ true\ x$. The scope of x is above the line.

Now for elimination rules. What information do we have given $A \to B\ true$? It turns out we actually have no information. If we don't know anything about

A then we don't know anything about this prop. So our elimination rule needs two pieces of information.

$$\frac{A \to B \; true \qquad A \; true}{B \; true}$$

6.2 Soundness

We can see how we derive this elimination rule. We can rewrite to

$$\frac{\begin{array}{c} A \; true : x \\ \vdots \\ B \; true \end{array} \qquad A \; true : e}{B \; true}$$

We can see that if we substitute e for x we will get our conclusion.

6.3 Completeness

Now we will expand our elimination rule to get our original introduction rule. We start with $A \to B \; true$ we expand to

$$\frac{\dfrac{A \to B \; true \qquad A \; true}{B \; true}}{A \to B \; true}$$

7 Disjunction: A small example

Let's introduce disjunction.

$$\frac{A \; true}{A \vee B \; true} \vee\text{-I}_1$$

$$\frac{B \; true}{A \vee B \; true} \vee\text{-I}_2$$

This is simple. What about the elimination rules?

You might be tempted to do

$$\frac{A \vee B \; true}{A \; true} \vee\text{-E}$$

This is dangerous! If we add this rule then we can prove any proposition! Here is an example,

$$\frac{\dfrac{A \to A \; true}{A \vee (A \to A) \; true} \vee\text{-I}_2}{A \; true} \; .$$

But since this is schematic, this proves any A is true. Even false! Here is the correct rule

$$\frac{A \lor B\ \mathit{true} \qquad \begin{array}{c} A\ \mathit{true} \\ \vdots \\ C\ \mathit{true} \end{array} \qquad \begin{array}{c} B\ \mathit{true} \\ \vdots \\ C\ \mathit{true} \end{array}}{A \lor B\ \mathit{true}} \lor\text{-E} \quad .$$

8 Misc

- Convention dictates that there is only one conclusion in an inference.
- Premises of a hypothetical proof has a scope limited to the inference rule.
- Hypothetical proofs can be turned into an implication.
- In linear logic, premises can only be used once, while in intuitionistic logic (this series) they can be used as many times as needed.
- Premises are ordered, in the sense that they have different computational representation, although they usually have the same meaning when read out (E.g. A is true and B is true v.s. B is true and A is true).
- You might be tempted to define the introduction rule using the swap rule. This would be unfortunately circular. It is important that intro rules only refer to the needed components.

9 Homeworks

1. Prove that the rules for disjunction are sound and complete.
2. Given the following elimination rule for unknown connector $\bullet$ find a valid intro rule

$$\frac{A \bullet B\ \mathit{true} \qquad \begin{array}{c} A\ \mathit{true} \\ \vdots \\ C\ \mathit{true} \end{array} \qquad \begin{array}{c} B\ \mathit{true} \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}} \bullet\text{-E} \quad .$$