Pfenning Proof Theory Lecture 3

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1 Logistics

- Implicitly, all "prove X" problems are "or show that it can't be done"

2 Exercises

1. Prove or refute (using SEQ)

   - $A \to (B \vee C) \dashv\vdash (A \to B) \vee (A \to C)$
   - $(A \to B) \to C \dashv\vdash (A \vee C) \wedge (B \to C)$

     For those that hold, extract a proof.

     (Frank tried hard to ensure that these are all clasically true, but not all are intuitionistically true.)
   - Give $A \cdot B$ left and right rules.

3 Propositions as Types Review

- Assign computational meaning to proofs -
- Assign logical meanings to programs

4 Classical Logic: More Problems

You may think that there is nothing interesting left to do in classical logic. We have done all of the usual operators. But actually there are some interesting classical logic avenues to explore.

4.1 S_4

S_4 is a modal logic. S_4 allows propositions to be true sometimes. $\Box A$ means that A is always true. For example $A \supset A$ is always true so the prop $\Box A \supset A$ holds. However in S_4 $A \rightarrow boxA$ would not necessarily hold because things can be true some of the time but not always.

This is analogous to quoted code (well typed syntax objects inside a programming language). $\Box A$ is like quote A. Run is like unquoting and running the quoted code. run : $\Box A \supset A$

4.2 Other Interesting Paths

Temporal logic is another interesting. Certain propositions only hold at specific times. This can be used for both linear logic, and for modeling running time of programs.

The diamond and circle modalities from modal logic correspond to monadic and strong monadic computation

At least two axes we can change: adding operators (modal logic), changing the presentation (e.g. combinatory logic has a different computational interpretation than what we have seen)

5 Sequent Calculus

Previously we use natural deduction to derive propositions. These hypothetical judgments may look something like this:

$$\begin{array}{c} A_1\ true, \ldots, A_n\ true \\ \vdots \\ C\ true \end{array}$$

$A_1;\ true, \ldots, A_n\ true$ are the assumptions and $C\ true$ is the conclusion. This is the main idea of sequent calculus. We will condense this into easier syntax:

A sequent is:

$A_1, \ldots, A_n \vdash C$ The assumptions $A_1, \ldots, A_n$ are called the **antecedents** and the conclusion is the **succedent**.

5.1 Left and Right Rules

In natural deduction we construct proofs by using elimination rules top down and introduction rules bottom up. In sequent calculus rules all go bottom up. Instead we have left rules that change the antecedents and right rules that change the succedent.

6 Sequent Conjunction

The right-rule for conjunction is:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R$$

Corresponding to:

$$\begin{array}{ccc} \Gamma & \Gamma & \Gamma \\ \vdots & \longrightarrow \vdots & \vdots \\ A \wedge B & A & B \end{array}$$

(1) (2) (3)

We can see that this means if we want to prove $A \wedge B$ we can split it into two proofs, one of A and one of B.

We also have the left rules

$$\frac{\Gamma, A \wedge B, A \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L_1$$

$$\frac{\Gamma, A \wedge B, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L_2$$

We can see from these left rules if given a proof of conjunction we can destruct and get both components.

7 Sequent Implication

We can do a right-rule for $\supset$:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset R$$

What does the elimination rule for $\supset$ do?

$$\Gamma, A \supset B \qquad \Gamma, A \supset B \qquad \Gamma, A \supset B, B \vdots \quad \rightarrow \quad \vdots \qquad \vdots C \qquad A \qquad C$$

Frank used to think it was the following, but this isn't right:

$$\Gamma, A, A \supset B \qquad \Gamma, A, A \supset B, B \vdots \quad \rightarrow \quad \vdots C \qquad C$$

This is incorrect because it has too strong an assumption. We can not assume we will be given an A. Instead we require a proof of A before we can destruct $A \supset B$. Aside on ordered logic: You could treat Γ as a list, and then be explicit about the order of propositions in it. This is ordered logic. We do not do this however. Assume that the order elements appear in Γ is inconsequential.

The correct left rule for $\supset$ is:

$$\frac{\Gamma, A \supset B \vdash A \qquad \Gamma, A \supset B, B \vdash C}{\Gamma, A \supset B \vdash C} \supset L$$

How do we know when we have finished a proof? We need an identity rule:

$$\frac{}{\Gamma, A \vdash A} ID$$

When doing a natural deduction proof, we typically use introduction rules from the bottom up and elimination rules from the top down. When they meet we have finished the proof. In sequent calculus, left rules correspond to the reverse of natural deduction elimination rules, and right rules correspond to introduction rules. They both go bottom up. So we know the proof is finished when all that is left are the antecedents i.e. we can only apply ID.

An example sequent calculus proof:

$$WRONG\ \cfrac{\cfrac{\cfrac{\cfrac{A \supset (B \supset C), A, B \wedge B, A, B \vdash A \qquad A \supset (B \supset C), A, B \wedge B, A, B \vdash C}{A \supset (B \supset C), A, B \wedge B, A \vdash C}}{A \supset (B \supset C), A \wedge B, A \vdash C}}{A \supset (B \supset C), A \wedge B \vdash C}}{A \supset (B \supset C) \vdash (A \wedge B) \supset C} \supset R$$

8 Back to Natural Deduction

In order to show that this sequent calculus corresponds to natural deduction, we're doing to annotate our sequent calculus rules with proof terms.

E.g. $M_1 : A_1, \ldots M_n : A_n \vdash N : C$

Here are our annotated rules:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \wedge B} \wedge R$$

We can see that just as in natural deduction conjunction corresponds to pairs.

$$\frac{\Gamma, M : A \wedge B, \pi_1 M : A \vdash N : C}{\Gamma, M : A \wedge B \vdash N : C}$$

$$\frac{\Gamma, M : A \wedge B, \pi_2 M : B \vdash N : C}{\Gamma, M : A \wedge B \vdash N : C}$$

We can then destruct the pairs with projection functions.

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x.M) : A \supset B} \supset R^x$$

Implication corresponds to functions.

$$\frac{\Gamma, M : A \supset B \vdash N : A \qquad \Gamma, M : A \supset B, MN : B \vdash P : C}{\Gamma, M : A \supset B \vdash P : C} \supset L$$

Elimination of functions is application.

$$\frac{}{\Gamma, M : A \vdash M : A} ID$$

- [ ] Example annotated with proof terms

We see how we can unroll from sequent into natural deduction. If this holds for all sequents then we have the following metatheorem: If for all of the antecedent A_i we can construct a term $M_i : A_i$ then we can construct a term for the succedent $N : C$.

If we can always go from a sequent to a well-typed proof term then we know it is in natural deduction, this is a proof of soundness.

9 Why Sequent Calculus?

Consider $\bot$. There is no way to introduce it so we have no right rule. The left rule is

$$\frac{}{\Gamma, \bot \vdash C} \bot L$$

There is no proof of falsehood! $\cdot \vdash \bot$ This means that sequent calculus is a consistent system. Since we know sequent is mappable to natural deduction natural deduction is also consistent.

It is easier to disprove certain things in sequent, if they can't be proved we know they also can't be proved in natural deduction.

Here are the rules for disjunction:

$$\frac{\cdot \vdash A}{\cdot \vdash A \vee B}$$

$$\frac{\cdot \vdash B}{\cdot \vdash A \vee B}$$

Suppose I wanted to prove LEM:

$$\cdot \vdash A \vee (A \supset \bot)$$

For arbitrary A, we can't prove this because we can't prove A or $A \supset \bot$.

Of course, we can also sometimes prove something is unprovable is by showing that it's unprovable in classical/boolean logic. (We could add $A \vee \neg A$ as an axiom, which would allow us to prove at least as much as without it.)

In response to a question about how/why this sequent calcluls is a bit different than e.g. linear logic:

We can refactor the rules to add an explicit "contraction rule"

$$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} CONTR$$

And then have the left rules remove the thing their acting upon from the antecedents

E.g.

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L$$