Pfenning Proof Theory Lecture 4

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1 Lecture 3 Review

- Defined left and right rules for Sequent Calculus
- Assign proof terms to Sequent derivations to show to relationship to natural deduction
- Proofs that apply intro from below and elim from above can have an equivalent sequent constructed

2 Completeness of Sequent Calculus

Last time we proved the soundess of the sequent calculus. Since we can construct proof terms for every sequent we know every sequent is in natural deduction and therefore it is sound. But what about completeness?

We know that every natural deduction proof that applies intro from below and elimination from above has a sequent. But if we can construct a natural deduction proof that doesn't follow these rules we are in trouble. Last time we showed that sequent calculus was consistent because there was no rule that could prove false.

$$\frac{STUCK}{\bullet \vdash \bot}$$

And since we assumed completeness we also have completeness in natural deduction. But why did we need that? Can we use a rule for a proof of false in natural deduction? Actually yes.

$$\frac{\dfrac{\vdots}{A \supset L} \qquad \dfrac{\vdots}{A}}{\bullet \vdash \bot}$$

This is a valid proof using the implication elimination rule. But it's using it bottom up which is why there is no sequent for it. But this proof is impossible to construct. Can we construct a valid proof that is also using elim bottom up?

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y : A}\, y}{(\lambda y.y) : A \supset A}\, \supset\text{-I}^y}{\lambda x.(\lambda y.y) : \top \supset (A \supset A)}\, \supset\text{-I}^x \qquad \dfrac{}{\langle\rangle : \top}\, \top\text{-I}}{(\lambda x.(\lambda y.y))\langle\rangle : A \supset A}\, \supset\text{-E}$$

This is a perfectly valid proof. We can introduce $\top$ without issue since it is always true. We can then use implication elimination bottom up. We can reduce this proof to the following.

$$\dfrac{\dfrac{}{y : A}\, y}{\lambda y.y : A \supset A}\, \supset\text{-I}^y$$

We can see that this reduction is simple substitution.

$$(\lambda x.(\lambda y.y))\langle\rangle \longrightarrow \lambda y.y$$

This proof term can be made into a sequent. So we see that an equivalent proof can be made into a sequent, but not the original. There are now two different paths we could take to prove sequent is complete.

1. We could attempt to prove that, like for the above example, all proofs can be reduced to a form that has a sequent.
2. We could add a rule to sequent and show that it allows more expansive terms.

We will take option 2, although we will discuss option 1 at the end.

3 Cut

The rules for sequent calculus are in figure 1. We want to add a rule that will allow more expansive terms. This way we can prove the sequent calculus complete. We will add the ability to have lemmas. Now it is allowed to prove a subgoal before the original goal. The rule is cut:

$$\dfrac{\Gamma \vdash A \qquad \Gamma, A \vdash C}{\Gamma \vdash C}\, \text{Cut}$$

We can then see cut corresponds to let binding as proof terms.

$$\dfrac{\Gamma \vdash M : A \qquad \Gamma, x : A \vdash N : C}{\Gamma \vdash \mathit{let}\ x = M\ \mathit{in}\ N : C}\, \text{Cut}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-R} \qquad \frac{\Gamma, \_, A \vdash C}{\Gamma, A \wedge B \vdash C} \wedge\text{-L}_1 \qquad \frac{}{\Gamma \vdash \top} \top\text{-R}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset\text{-R} \qquad \frac{\Gamma, \_, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge\text{-L}_2 \qquad \frac{\Gamma, \_, \vdash A \qquad \Gamma, \_, B \vdash C}{\Gamma, A \supset B \vdash C} \supset\text{-L}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee\text{-R}_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee\text{-R}_2 \qquad \frac{\Gamma, \_, A \vdash C \qquad \Gamma, \_, B \vdash C}{\Gamma, A \vee B \vdash C} \vee\text{-L}$$

Figure 1: Rules for Sequent Calculus

So now we have a rule that will allow all natural deduction proofs to have a corresponding sequent. But things are not as good as they seem. Now we have broken the property we wanted for sequent calculus! The whole point was to only allow rules that go bottom up! But all is not lost! We will now prove that cut can always be removed, anything that relies on cut to be proved can be proved without it.

4 Cut Elimination

From a practical standpoint this makes sense. Cut is let expressions, let expressions can always be removed by inlining. Computationally this may increase the amount of work done, but the result will be the same.

Dashed line: there exists a derivation, but not by a rule but more by analysing the proof.

Theorem 4.1 (Gentzen 1935)**.** *Cut is* ***admissible****. That is, if* $\Gamma \vdash A$ *and* $\Gamma, A \vdash C$ *then* $\Gamma \vdash C$*. Admissible is different from derivable. If something is derivable it can be proved by the rules. Admissible means that it has been shown to hold for all rules. If you add more rules to the system derivable things are fine, but admissible things must be shown for any added rules.*

$$\dfrac{\dfrac{\mathcal{D}}{\Gamma \vdash A} \qquad \dfrac{\mathcal{E}}{\Gamma, A \vdash B}}{\Gamma \vdash B} \ \text{Cut}$$

Proof. By nested induction, first on A, second on $\mathcal{D}$ **and** $\mathcal{E}$. In other words, lexicographical induction $\langle A, [\mathcal{D}, \mathcal{E}] \rangle$. □

$$\frac{\Gamma \vdash x : A \qquad \Gamma \vdash y : B}{\Gamma \vdash \langle x, y \rangle : A \wedge B} \wedge\text{-R} \qquad\qquad \frac{\Gamma, \_, \pi_1(M) : A \vdash N : C}{\Gamma, M : A \wedge B \vdash N : C} \wedge\text{-L}_1$$

$$\frac{}{\Gamma \vdash \langle\rangle : \top} \top\text{-R} \qquad\qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \supset B} \supset\text{-R}$$

$$\frac{\Gamma, \_, \pi_2(M) : B \vdash N : C}{\Gamma, y = M : A \wedge B \vdash N : C} \wedge\text{-L}_2$$

$$\frac{\Gamma, \_, \vdash N : A \qquad \Gamma, \_, (PN) : B \vdash M : C}{\Gamma, P : A \supset B \vdash M : C} \supset\text{-L} \qquad\qquad \frac{\Gamma \vdash \iota_1(M) : A}{\Gamma \vdash M : A \vee B} \vee\text{-R}_1$$

$$\frac{\Gamma \vdash \iota_2(M) : B}{\Gamma \vdash M : A \vee B} \vee\text{-R}_2$$

$$\frac{\Gamma, \_, x : A \vdash N_1 : C \qquad \Gamma, \_, y : B \vdash N_2 : C}{\Gamma, M : A \vee B \vdash case(M)\{\iota_1 x \to N_1 \| \iota_2 y \to N_2\} : C} \vee\text{-L}$$

Figure 2: Rules for Sequent Calculus

5 Normal and Neutral terms

Earlier we mentioned that we could also prove completeness from the other direction by showing all natural deduction proofs could be reduced to a form that could be made sequent. These are the neutral and normal forms. In a sequent $R_1 : A_1, \vdots, R_n : A_n \vdash N : C$ the neutral terms correspond to the antecedents $R_1, \vdots R_n$ and normal terms correspond to the succedent N. In order to see their forms we look again at the rules for sequent calculus. Figure 2 contains the sequent calculus rules annotated with their proof terms. We can simply read off the possibilities for antecedents $R = \pi_1(x), \pi_2(x), \iota_1(x), \iota_2(x), x, (RN)$ these are the possibilities for neutral terms. We can do the same for normal terms $N = \lambda x.N, \langle\rangle, \langle N, M\rangle, R, case(R)\{\iota_1 x \to N_1 \| \iota_2 y \to N_2\}, case(R)()$. Reducing terms to their neutral or normal forms is a different way to show the completeness of sequent, every term can be reduced to one of those and they all have corresponding sequents. However that is a harder proof than the way we did it.

6 Exercises :

1. Show the cases for admissibility of cut where:
   (a) $\vee$-R meets $\vee$-L
   (b) $\vee$-L meets anything