OPLSS 2023: Logical Relations [*]

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Abstract

These are lecture notes for the Logical Relations course at OPLSS 2023. The notes cover an introduction to logical relations, along with detailed examples, namely: normalization and type safety for simply-typed lambda calculus (STLC), step-indexed logical relation for recursive types, and a brief look at binary logical relations for parametric polymorphism.

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1 Introduction

Logical relations are a prolific and highly useful proof method that can be used to prove a wide range of properties about programs and languages. For instance, logical relations can help to prove:

- termination of well-typed programs in a language such as the simply-typed lambda calculus (STLC);
- type soundness/safety for languages like the STLC;
- program equivalence, which can take many forms:
  - Verifying that an optimized algorithm or implementation is equivalent to a simpler naive one;
  - Demonstrating the correctness of compiler optimizations or transformations;

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- Showing *representation independence* [Mit86], that is, that the implementation underlying a given interface does not influence the behavior of the client;
    - For instance, a stack interface may be implemented using an array or a linked list, but this should be indistinguishable to the users of the interface. To show representation independence for the stack interface, we would show that a program that uses the array implementation is equivalent to a program that uses the linked list implementation.
- Conceptualizing parametric polymorphism and corresponding free theorems [Wad89] as a relation loosely described as "related inputs go to related outputs;"
- Proving noninterference for security-typed languages: showing that two runs of a given program are equivalent on low-security outputs for any variation on high-security (e.g. confidentiality) program data

and much more!

In recent work, logical relations been used to prove the soundness of different logics. Cross-language logical relations are used to reason about compilation, foreign function interfaces or multi-language semantics.

The terms *logical predicate* and *unary logical relation* are synonymous, as are the terms *logical relation* and *binary logical relation*. Both terms are used in the literature and common parlance. Logical predicates $P_\tau(e)$, are sets, and reason about a single program. Program termination and type safety and soundness are logical predicates. On the other hand, logical relations $R_\tau(e_1, e_2)$ are binary relations, and can be used to capture program equivalences more broadly.

2 Normalization of STLC

In this section, we show an example of a logical relation for proving normalization of STLC.

2.1 Formalization of STLC

We recall the syntax and operational semantic of STLC in Figure 1. We consider a slightly variant of STLC, with the base type of booleans, and the if-then-else expression.

$$\begin{aligned}
\tau &::= \mathsf{bool} \mid \tau \to \tau \\
e &::= x \mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ e\ \mathsf{then}\ e\ \mathsf{else}\ e \mid \lambda x : \tau.\ e \mid e\ e \\
v &::= \mathsf{true} \mid \mathsf{false} \mid \lambda x : \tau.\ e \\
E &::= [] \mid \mathsf{if\ E\ then}\ e\ \mathsf{else}\ e \mid \mathrm{E}\ e \mid v\ \mathrm{E}
\end{aligned}$$

$$\textsc{E-IfThenElseTrue}\quad \frac{}{\mathsf{if\ true\ then}\ e_1\ \mathsf{else}\ e_2 \mapsto e_1} \qquad \textsc{E-IfThenElseFalse}\quad \frac{}{\mathsf{if\ false\ then}\ e_1\ \mathsf{else}\ e_2 \mapsto e_2} \qquad \textsc{E-App}\quad \frac{}{(\lambda x : \tau.\ e)\ v \mapsto e[v/x]}$$

$$\textsc{E-Step}\quad \frac{e \mapsto e'}{E[e] \mapsto E[e']}$$

Figure 1: Call-by-value small step semantics of STLC with evaluation context. $\mapsto$ is the head reduction relation (also called *primitive reduction*).

The operational semantic implements a *call-by-value* evaluation order, meaning that the argument of a function needs to be evaluated into a value before applying the function. We note $e[v/x]$ the substitution of the variable x by the value v in the term e. We write $e \mapsto e'$ to say that e reduces to e' in one computation step. We write $e \mapsto^* e'$ for the transitive and reflexive closure of the reduction relation.

Figure 2 shows the typing judgment rules. The relation $\Gamma \vdash e : \tau$ means that x is typed by τ under the typing context Γ. The typing Γ is a mapping from variable to their type. We do not go through the details of the typing rules, as they are fairly classic. In this paper, we work using implicit α-renaming, that is the free variables (in Γ) do not overlap with bounded variables.

$$\boxed{\Gamma \vdash e : \tau}$$

$$\text{T-TRUE}\ \frac{}{\Gamma \vdash \text{true} : \text{bool}} \qquad \text{T-FALSE}\ \frac{}{\Gamma \vdash \text{false} : \text{bool}} \qquad \text{T-VAR}\ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \text{T-IFTHENELSE}\ \frac{\Gamma \vdash e : \text{bool} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau}$$

$$\text{T-ABS}\ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2} \qquad \text{T-APP}\ \frac{\Gamma \vdash e_1 : \tau_2 \to \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau_1}$$

Figure 2: Typing judgments of STLC.

Then, we formally define normalization (N) of STLC, meaning that any well-typed closed terms actually terminates. We introduce the notation $e \Downarrow v$ to mean that the term e evaluates in to value v, that is $e \mapsto^* v$, and the notation $e \Downarrow$ to mean that e evaluates to some value , that is $\exists v. e \mapsto^* v$.

Theorem (Normalization)**.** *For all terms e, if $\cdot \vdash e : \tau$, then $e \Downarrow$.*

2.2 First attempt: Proving STLC is normalizing by induction on typing derivations

To see why we might want to use logical relations, let's first try to prove this normalization theorem2.1 naively, and understand where we get stuck.

Proof. Suppose $\cdot \vdash e : \tau$. Show $e \Downarrow$. We proceed by induction on the typing derivation:
- Case T-TRUE $\frac{}{\cdot \vdash \text{true} : \text{bool}}$: true is already a value, so it is terminated.
- Case T-FALSE $\frac{}{\cdot \vdash \text{false} : \text{bool}}$: false is already a value, so it is terminated.
- Case T-VAR $\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$: by induction hypothesis, we get that $\cdot(x) = \tau$, which means that this case is vacuously true (because our context is empty).
- Case T-ABS $\frac{x : \tau_1 \vdash e : \tau_2}{\cdot \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2}$: $\lambda x : \tau_1.\ e$ is already a value, so it is terminated.
- Case T-IFTHENELSE $\frac{\cdot \vdash e : \text{bool} \qquad \cdot \vdash e_1 : \tau \qquad \cdot \vdash e_2 : \tau}{\cdot \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau}$: From the inductive hypothesis, we know that $e \Downarrow$, $e_1 \Downarrow$ and $e_2 \Downarrow$. Since $e \Downarrow v$ for some v, by inspecting our reduction semantics, we know that if e then e_1 else $e_2 \mapsto^*$ if v then e_1 else e_2. Thanks to the canonical forms lemma, we know that the only values of type bool are true and false. By case destruction over v, either if e then e_1 else $e_2 \mapsto^*$ if true then e_1 else $e_2 \mapsto^* e_1 \Downarrow$, or if e then e_1 else $e_2 \mapsto^*$ if false then e_1 else $e_2 \mapsto^* e_2 \Downarrow$.
- Case T-APP $\frac{\cdot \vdash e_1 : \tau_2 \to \tau \qquad \cdot \vdash e_2 : \tau_2}{\cdot \vdash e_1\ e_2 : \tau}$: From the inductive hypothesis, we know that $e_1 \Downarrow$ and $e_2 \Downarrow v_2$. The values of type $\tau_2 \to \tau$ are lambda expressions, so since $e_1 \Downarrow$, we know $e_1 \Downarrow \lambda x : \tau_2.\ e'$, so $e_1\ e_2 \mapsto^* (\lambda x : \tau_2.\ e')\ e_2 \mapsto^* e'[v_2/x]$. But we don't know anything about whether e' terminates, so we are stuck and cannot prove this case. ■

The main insight is, the induction hypothesis is too weak: the typing judgment does not tell us anything about the value produced by the normalization. To solve this issue, we will define a logical relation that precisely strengthens the induction hypothesis, allowing us to know more about the value produced by the evaluation.

2.3 Principles for defining a logical relation

The logical relation is a predicate $P_\tau(e)$ over an expression, indexed by a type. When defining a logical relation $P_\tau(e)$, there are three principles we typically follows:

- The logical relation should contain well-typed terms, ie. $\cdot \vdash e : \tau$

- The property of our interest P should be baked in the logical relation (i.e., normalization)
- The property of our interest P should be preserved by the elimination forms[1] for the corresponding type τ. Intuitively, it means that we have a way to continue after consuming a value of type τ.

We will now see how to construct a logical relation more concretely to prove that STLC is normalizing.

2.4 Logical relation for normalization of STLC

We define the normalization logical predicate by induction on STLC types, following the three principles previously defined:

$$\begin{array}{rcl} N_{\texttt{bool}}(e) & \triangleq & \cdot \vdash e : \text{bool} \quad \wedge \quad e \Downarrow \\ N_{\tau_1 \to \tau_2}(e) & \triangleq & \cdot \vdash e : \tau_1 \to \tau_2 \quad \wedge \quad e \Downarrow \quad \wedge \quad \forall e'.\ N_{\tau_1}(e') \implies N_{\tau_2}(e\ e') \end{array}$$

Comparing with the aforementioned principles, we see that the left-most conjuncts in each case ensure the logical relation contains only well-typed terms. Similarly, the next conjunct in both cases ensures the logical relation only contains terms satisfying the desired property, ie. normalization. Finally, we need to ensure the property of interest is preserved by elimination forms, which is handled by the third conjunct of the $\tau_1 \to \tau_2$ case. Interestingly, we do not need to say anything about the elimination form of bool, which is due to its positive polarity. As a rule of thumb, types with positive polarity do not need any extra conjuncts, while types with negative polarity do.

From here, we can split the proof of normalization into two steps:

(A) For all terms e, if $\cdot \vdash e : \tau$ then $N_\tau(e)$

(B) For all terms e, if $N_\tau(e)$ then $e \Downarrow$

Step (B) follows quickly from the definition of the logical predicate N_τ. By induction on τ, we have two cases to consider, and in both cases $e \Downarrow$ follows directly from the definition of $N_\tau(e)$. One might remark that, it actually comes from the second principle: the property of our interest (normalization) is in the definition of the logical relation.

Step (A) is not so straightforward however due to the case of lambda abstraction. Suppose that $\cdot \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2$ so in particular we know $x : \tau_1 \vdash e : \tau_2$. To prove (A), we now need to show that $\cdot \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2$ which holds by assumption. We must also show that $\lambda x : \tau_1.\ e \Downarrow$, which holds as a lambda term is already a value. The final step is to prove that for any term e', if $N_{\tau_1}(e')$ then $N_{\tau_2}(\lambda x : \tau_1.\ e\ e')$. At this point we find ourselves stuck, since e is not a closed term and thus we cannot proceed by induction on its typing derivation. This leads us to generalising step (A) to allow for open terms:

(A') For all contexts Γ, substitutions γ and terms e if $\Gamma \vdash e : \tau$ and $\gamma \vDash \Gamma$ then $N_\tau(\gamma(e)))$

(B) For all terms e if $N_\tau(e)$ then $e \Downarrow$

The substitution $\gamma = \{x_1 \mapsto v_1, \ldots, x_n \mapsto v_n\}$ cannot be any arbitrary substitution. Intuitively, we say that $\gamma \vDash \Gamma$ (read γ satisfies the typing environment Γ) if they have the same domain and if all the inputs respects the property of our interest (in our case, normalization). This is required because a value in the context can be a lambda abstraction, which ends up with the same issue that we had in the first place.

$$\gamma \vDash \Gamma \triangleq \text{dom}(\gamma) = \text{dom}(\Gamma) \wedge (\forall x \in \text{dom}(\Gamma).\ N_{\Gamma(x)}(\gamma(x)))$$

Then, we define some auxiliary lemmas. In proving step (A') we will make use of two lemmas. The first is the substitution lemma, which will let us go from well typed-open terms to well-typed closed terms. The interested reader can check [Pie02, p. 9.3.8]

Lemma (Substitution Lemma)**.** *Substitution preserves types, that is if* $\Gamma \vdash e : \tau$ *and* $\gamma \vDash \Gamma$ *then* $\cdot \vdash \gamma(e) : \tau$.

The second lemma is the reduction lemma, which will be important for the lambda case when we will need to reduce terms.

Lemma (Reduction Lemma)**.** *The normalization logical predicate respects reduction, that is if* $\cdot \vdash e : \tau$ *and* $e \mapsto e'$ *then* $N_\tau(e) \iff N_\tau(e')$.

With these lemmas, we are ready to prove (A').

Lemma (Generalized normalization)**.** $\forall\ \Gamma, \gamma, e,\ \tau.\ \Gamma \vdash e : \tau \wedge \gamma \vDash \Gamma \Rightarrow N_\tau(\gamma(e))$

[1]The introduction forms creates a value of type τ, while the elimination form consumes a value of type τ

Proof. Suppose $\Gamma \vdash e : \tau$ and $\gamma \vDash \Gamma$. Show $\Rightarrow N_\tau(\gamma(e))$. We proceed by induction on the typing derivation:
- Case T-True $\dfrac{}{\Gamma \vdash \text{true} : \text{bool}}$: We need to show $N_{\text{bool}}(\gamma(\text{true}))$. By definition of the substitution, it is equivalent to show $N_{\text{bool}}(\text{true})$. By unfolding the definition of N_{bool}, we need to show $\cdot \vdash \text{true} : \text{bool}$ and true $\Downarrow$. Both goals are trivial.
- Case T-False $\dfrac{}{\Gamma \vdash \text{false} : \text{bool}}$: similarly to the T-True case.
- Case T-Var $\dfrac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$: We need to show $N_\tau(x)$. Because $\gamma \in \Gamma$, we know that $x \in \Gamma$ and $N_{\Gamma(x)}(x)$, which is equivalent to $N_\tau(x)$.
- Case T-IfThenElse $\dfrac{\Gamma \vdash e : \text{bool} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau}$: Proof is left as an exercise for the reader. Another exercise for the reader is to type up the proof and send it to us.
- Case T-App $\dfrac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau_1}$: We need to show $N_\tau(\gamma(e_1\ \ e_2))$, which is equivalent to showing $N_\tau(\gamma(e_1)\ \gamma(e_Z))$. Because τ is arbitrary here, we should read the induction hypothesis. By induction hypothesis, we know that $N_{\tau_2 \to \tau}(e_1)$. By definition, it means that $\forall e',\ N_{\tau_1}(e') \Rightarrow N_{\tau_2}(\gamma(e_1\ e'))$. We can instantiate e' and τ_1 with $\gamma(e_2)$ and τ. Thanks to the other induction hypothesis, we know that $N_{\tau_1}(\gamma(e_2))$. It thus gives us $N_\tau(\gamma(e_1)\ \gamma(e_2))$, which is what we are trying to prove.

- Case T-Abs $\dfrac{x : \tau_1, \Gamma \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2}$: We suppose $\Gamma \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2$ and $\gamma \vDash \Gamma$. We want to show that $N_{\tau_1 \to \tau_2}(\gamma(\lambda x : \tau_1.\ e))$. By unfolding the definition of N, there are three properties to prove:
(1) $\cdot \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2$
(2) $\lambda x : \tau_1.\ e \Downarrow$
(3) $\forall e'.\ N_{\tau_1}(e') \Rightarrow N_{\tau_2}(\gamma(\lambda x : \tau_1.\ e)\ e')$

(1) follows directly from the substitution lemma 2.4. (2) is trivial, because $\lambda x : \tau_1.\ e$ is already a value. Let us prove (3).

Suppose e' such that $N_{\tau_1}(e')$. We need to show that $N_{\tau_2}(\gamma(\lambda x : \tau_1.\ e)\ e')$. We can push the substitution γ inside the lambda, which is then equivalent to show $N_{\tau_2}((\lambda x : \tau_1.\ \gamma(e))\ e')$.

From $N_{\tau_1}(e')$, we also know that it exists a value v' such that $e' \mapsto^* v'$. Using the backward reduction lemma 2.4, it then actually suffices to prove $N_{\tau_2}((\lambda x : \tau_1.\ \gamma(e))\ v)$. Taking another step, we eventually need to prove $N_{\tau_2}(\gamma(e)[v'/x])$.

We did not use the induction hypothesis yet, let's take a look at it. The induction hypothesis is: $\forall \gamma'.\ \Gamma, x : \tau \vdash e : \tau_2 \wedge \gamma' \vDash \Gamma, x : \tau \Rightarrow N_{\tau_2}(\gamma'(e))$. We notice the typing environment $\Gamma, x : \tau$. It is important to understand what is happening when applying the induction principle, because this is a common case of error. Here, the induction hypothesis says that, giving a substitution γ' that satisfies the environment $\Gamma, x : \tau$, we get $N_{\tau_2}(\gamma'(e))$. In particular, we get to pick the γ'. More specifically, we need to pick a value $v_?$ such that $\gamma' = \gamma[x \mapsto v_?]$ and $N_\tau(v_?)$. We do have a candidate: v'.

We already know that $N_{\tau_1}(e')$ and $e' \mapsto^* v'$, and using the forward propagation 2.4, we know that $N_{\tau_2}(v')$. As such, we pick $\gamma' \triangleq \gamma[x \mapsto v']$. We can thus instantiate the induction hypothesis, giving us $N_{\tau_1}(\gamma'(e'))$, which is equivalent to $N_{\tau_2}(\gamma[x \mapsto v'](e))$. By definition of the substitution, it finally give us $N_{\tau_2}(\gamma(e[v'/x]))$, which is precisely what we wanted to prove. □

One might notice that, while the naive proof is easy for the abstraction case and gets stuck for the application case, this is the other way for this proof: the application case is easy, while the abstraction case is hard. This is usually the case with logical relations.

3 Type safety of STLC

Type safety is informally defined by the slogan "well-typed programs do not go wrong", often attributed to Robin Milner. Alternatively, one can informally describe type safety as "well-typed programs do not get *stuck*." What this means is that, when a well-typed program runs, there won't be a nonsensical operation, such as trying to apply a string to a number (e.g. "foo" 5).

Formally, we say that a term e is safe if:

$$\text{safe}(e) \triangleq \forall e'.\ e \mapsto^* e' \Rightarrow \text{val}(e') \vee (\exists e''.\ e' \mapsto e'')$$

.

We define type safety as follows:

Theorem (Type safety)**.** *If* $\cdot \vdash e : \tau$*, then* $\mathrm{safe}(e)$*.*

Typically, type safety is proven using the progress and preservation lemmas:

Lemma (Progress)**.** *If* $\cdot \vdash e : \tau$ *then either* $\mathrm{val}(e)$ *or* $\exists e'$ *such that* $e \mapsto e'$*.*

Lemma (Preservation)**.** *If* $\cdot \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot \vdash e' : \tau$*.*

The progress and preservation lemmas refer to a single reduction step. To prove type safety, we use induction on the number of reduction steps and then apply the two lemmas.

3.1 Logical relation for type safety of STLC

To prove type safety of STLC using logical relations, we will define a logical relation on values ($\mathcal{V}[\![\tau]\!]$) and expressions ($\mathcal{E}[\![\tau]\!]$) separately by induction on the structure of types:

$$\begin{array}{rcl} \mathcal{V}[\![\mathrm{bool}]\!] & \triangleq & \{\mathrm{true}, \mathrm{false}\} \\ \mathcal{V}[\![\tau_1 \to \tau_2]\!] & \triangleq & \{\lambda x : \tau_1.\ e \mid \forall v \in \mathcal{V}[\![\tau_1]\!].\ e[v/x] \in \mathcal{E}[\![\tau_2]\!]\} \\ \mathcal{E}[\![\tau]\!] & \triangleq & \{e \mid \forall e'.\ e \mapsto^* e' \wedge \mathrm{irred}(e') \Rightarrow e' \in \mathcal{V}[\![\tau]\!]\} \end{array}$$

where

$$\mathrm{irred}(e) \triangleq \nexists e'.\ e \mapsto e'$$

We do a distinction between the value and the expression relation for clarity: this way, the reduction relation is only used in the expression relation, and the value relation only talks about the values.

We can check that $\mathcal{V}[\![\tau]\!]$ ("values that behave like the type τ") and $\mathcal{E}[\![\tau]\!]$ ("expressions that behave like the type τ") are well-founded relations on the structure of types. $\mathcal{V}[\![\tau_1 \to \tau_2]\!]$ refers to itself in its definition, which may seem problematic. However, it refers to $\mathcal{V}[\![\tau_1]\!]$, which is $\mathcal{V}[\![-]\!]$ for a smaller type (τ_1) than $\tau_1 \to \tau_2$. It also refers to $\mathcal{E}[\![\tau_2]\!]$, which again is $\mathcal{E}[\![-]\!]$ for a smaller type (τ_2) than $\tau_1 \to \tau_2$. Similarly, the definition of $\mathcal{E}[\![-]\!]$ refers to $\mathcal{V}[\![-]\!]$ for the same type. So those two relations are well-founded[2].

Now that we have defined our logical relations, we can use them to prove type safety for STLC. Similar to normalization, the proof of type safety for STLC can be split into two steps:

(A) For all terms e if $\cdot \vdash e : \tau$ then $\cdot \vDash e : \tau$

(B) For all terms e if $\cdot \vDash e : \tau$ then $\mathrm{safe}(e)$

Note that $\cdot \vDash e : \tau$ has not been defined yet. We need to do a bit more work to define it, but it is essentially equivalent to $e \in \mathcal{E}[\![\tau]\!]$.

To prove (B), we assume that $\cdot \vDash e : \tau$, so for all terms e' if $e \mapsto^* e'$ and e' is irreducible, then $e' \in \mathcal{V}[\![\tau]\!]$. To prove $\mathrm{safe}(e)$, we fix some e' and suppose $e \mapsto e'$, so we must show that either e' is a value or $\exists e''.e' \mapsto e''$. Now, either there exists some e'' such that $e' \mapsto e''$ in which case $\mathrm{safe}(e)$ or there exists no such e''. In the latter case, e' is irreducible and as $e \in \mathcal{E}[\![\tau]\!]$ we know that $e' \in \mathcal{V}[\![\tau]\!]$ and hence must be a value, so $\mathrm{safe}(e)$ holds in this case too.

Similar to the proof for normalization of STLC, we prove (A) as a corollary to the fundamental property of the type safety logical relation, which is a generalization of the statement for open terms. Before we can state this theorem, we must introduce the notion of semantic well-typedness. First, we give a definition of substitutions into a context that respect type safety, which is done by induction on the structure of contexts:

$$\mathcal{G}[\![\cdot]\!] \triangleq \{\emptyset\}$$
$$\mathcal{G}[\![\Gamma, x : \tau]\!] \triangleq \{\gamma[x \mapsto v] \mid \gamma \in \mathcal{G}[\![\Gamma]\!] \wedge v \in \mathcal{V}[\![\tau]\!]\}$$

Then, the definition of semantic well-typedness is that for any substitution γ that satisfies the typing context Γ, the substituted term $\gamma(e)$ is in the expression relation for the type τ. Formally,

$$\Gamma \vDash e : \tau \triangleq \forall \gamma \in \mathcal{G}[\![\Gamma]\!], \gamma(e) \in \mathcal{E}[\![\tau]\!]$$

.

[2]This is not always the case. For instance, we will see later that if our language has recursive types, we cannot define the relations by induction on type anymore.

Theorem (Fundamental Property/Basic Lemma of the Logical Relation $\mathcal{E}[\![-]\!]$)**.** *If a term is syntactically well-typed then it is semantically well-typed, that is, if* $\Gamma \vdash e : \tau$ *then* $\Gamma \vDash e : \tau$*.*

Proof. Suppose $\Gamma \vdash e : \tau$. We have to show $\Gamma \vDash e : \tau$. We proceed by induction on the typing judgment.

- Case T-True $\dfrac{}{\Gamma \vdash \text{true} : \text{bool}}$: given a substitution $\gamma \in \mathcal{G}[\![\Gamma]\!]$, we must show that $\gamma(\text{true}) = \text{true} \in \mathcal{E}[\![\text{bool}]\!]$. Since true is already irreducible, it is enough to show that true $\in \mathcal{V}[\![\text{bool}]\!]$, which is just the definition.
- Case T-False $\dfrac{}{\Gamma \vdash \text{false} : \text{bool}}$: similarly to the above case, we have to show that $\gamma(\text{false}) \in \mathcal{E}[\![\text{bool}]\!]$. To show this it suffices to prove false $\in \mathcal{V}[\![\text{bool}]\!]$ and this holds by definition.
- Case T-Var $\dfrac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$: as the set $\mathcal{G}[\![\Gamma]\!]$ was defined inductively, we now proceed by induction on the structure of Γ. The case of the empty context cannot come up as the context contains at least the variable x, so we are left with two cases. If $\Gamma \triangleq \Gamma', x : \tau$, then since $\gamma \in \mathcal{G}[\![\Gamma', x : \tau]\!]$ we also know that $\gamma(x) \in \mathcal{V}[\![\tau]\!]$. As values are irreducible this means that $\gamma(x) \in \mathcal{E}[\![\tau]\!]$. The second case has $\Gamma \triangleq \Gamma', y : \tau'$ with y a distinct variable from x and $\gamma \triangleq \gamma'[y \mapsto v]$. In this case we know that we have $\Gamma' \vdash x : \tau$ and $\gamma' \in \mathcal{G}[\![\Gamma']\!]$ since $\gamma \in \mathcal{G}[\![\Gamma]\!]$, but by the inductive hypothesis we get that $\Gamma' \vDash x : \tau$ so $\gamma(x) = \gamma'(x) \in \mathcal{V}[\![\tau]\!]$ and hence in $\mathcal{E}[\![\tau]\!]$.
- Case T-IfThenElse $\dfrac{\Gamma \vdash e : \text{bool} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau}$: to prove this case, we must inspect the reduction trace to decide which inductive hypothesis to use. This is a very important skill to have and as such we leave this case as a *useful* exercise to the reader.
- Case T-App $\dfrac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau_1}$: the case for application follows similarly to the proof of normalisation and is left as a *fun* exercise for the reader.
- Case T-Abs $\dfrac{x : \tau_1, \Gamma \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2}$:

We need to show $\Gamma \vDash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2$. By unfolding the definition of semantic typing, we suppose $\gamma \in \mathcal{G}[\![\Gamma]\!]$, and we need to show $\gamma(\lambda x : \tau_1.\ e) \in \mathcal{E}[\![\tau_1 \to \tau_2]\!]$. We push the substitution γ under the lambda. It thus remains to show that $\lambda x : \tau_1.\ \gamma(e) \in \mathcal{E}[\![\tau_1 \to \tau_2]\!]$.

By unfolding the definition of $\mathcal{E}[\![\tau_1 \to \tau_2]\!]$, suppose that we have an e' such that $(\lambda x : \tau_1.\ \gamma(e)) \mapsto^* e'$ and irred(e'), and we then need to show $e' \in \mathcal{V}[\![\tau_1 \to \tau_2]\!]$. One might notice that the operational semantic actually takes no step, which means that $e' = \lambda x : \tau_1.\ \gamma(e)$. As such, we need to show that $\lambda x : \tau_1.\ \gamma(e) \in \mathcal{V}[\![\tau_1 \to \tau_2]\!]$.

Let us take a look at the definition of $\mathcal{V}[\![\tau_1 \to \tau_2]\!]$. Suppose $v \in \mathcal{V}[\![\tau_1]\!]$, and show that $\gamma(e)[v/x] \in \mathcal{E}[\![\tau_2]\!]$.

By induction hypothesis, we have $\Gamma, x : \tau_1 \vDash e : \tau_2$. By unfolding the definition of semantic typing, we can provide a certain γ' such that $\gamma' \vDash \Gamma, x : \tau_1$. We can instantiate γ' with $\gamma[x \mapsto v]$, because (by definition of satisfaction of the environment) $\gamma \vDash \Gamma$ and $v \in \mathcal{V}[\![\tau_1]\!]$. Finally, the induction hypothesis gives us $\gamma'(e) \in \mathcal{E}[\![\tau_2]\!]$, which is equivalent to $\gamma[x \mapsto v](e) \in \mathcal{E}[\![\tau_2]\!]$. By definition of substitution, it is then equivalent to $\gamma(e[v/x]) \in \mathcal{E}[\![\tau_2]\!]$, which is exactly what we wanted to prove. □

4 A logical relation for type safety with recursive types in STLC

4.1 The purpose of recursive types

Recursive types can be used to capture potentially infinite data structures such as lists, streams, and trees, among others. The addition of recursive types to a language permit non-terminating programs to be type checked.

To demonstrate the utility of recursive types, let us examine the Ω combinator from the untyped lambda calculus, which loops infinitely:

$$\Omega = (\lambda x.xx)(\lambda x.xx)$$

If we try to apply the first term to the second, we step to the Ω combinator again. Now, let's suppose we tried to type this term in the simply-typed lambda calculus:

$$(\lambda x :?.xx)(\lambda x :?.xx)$$

In one appearance of x in the term $(\lambda x.xx)$, it is being applied to itself, so we would expect its type to be something like $\tau_1 \to \tau_2$. However, the x supplied as argument to the function x would not be of the correct input type. We will soon see how recursive types can be helpful in typing the Ω combinator.

Consider the recursive type describing a tree (here written using OCaml syntax):

```
type tree = Leaf | Node of int * tree * tree
```

We can rewrite this definition using the unit type 1 to represent tree leaves:

$$\text{type } \mathit{tree} = 1 + (\text{int} \times \mathit{tree} \times \mathit{tree})$$

tree appears in its own definition, which prevents us from inspecting it further. For the sake of clarity, we will use the type variable α to describes the type tree.

Note that we can *unfold* the definition of a tree to describe a tree with a larger depth by substituting the definition of a tree within the definition.

$$1 + (\text{int} \times \alpha \times \alpha) = 1 + (\text{int} \times (1 + (\text{int} \times \alpha \times \alpha)) \times (1 + (\text{int} \times \alpha \times \alpha)))$$

A fixpoint is a function f such that $x = f(x)$ for all $x \in \mathit{dom}(f)$. Given the following recursive function, we will define a fixpoint for it, namely some F such that $\mathit{tree} = F(\mathit{tree})$:

$$F = \lambda\alpha :: \text{type}.\ 1 + (\text{int} \times \alpha \times \alpha)$$

We will write the fixpoint of F as follows, and refer to it as the *fixed point type constructor* for α:

$$\mu\alpha.F(\alpha) = F(\mu\alpha.F(\alpha))$$

We will substitute τ for $F(\alpha)$; note that α is free to appear in τ:

$$\mu\alpha.\tau = F(\mu\alpha.\tau)$$

Note that we can rewrite the right-hand side straightforwardly, since $F(\alpha) = \tau$:

$$\mu\alpha.\tau = \tau[\mu\alpha.\tau/\alpha]$$

Intuitively, the left-hand side of this equation represents the fixpoint for τ, which in the case of trees, represents our initial definition of the tree type. The right-hand side represents the *unfolded* definition of trees that we saw afterwards. Indeed, going from the left-hand side of the above equation to the right-hand side represents *unfolding* a recursive type, while going the opposite way represents *folding* a recursive type.

4.2 Formalization of STLC with recursive types

We extend the STLC grammar with recursive types.

$$\begin{gathered}
\tau ::= 1 \mid \text{bool} \mid \tau \to \tau \mid \alpha \\
e ::= x \mid \langle\rangle \mid \text{true} \mid \text{false} \mid \lambda x.\ e \mid e\ e \mid \text{fold } e \mid \text{unfold } e \\
v ::= \langle\rangle \mid \text{true} \mid \text{false} \mid \lambda x : \tau.\ e \mid \text{fold } v \\
E ::= [] \mid \text{if E then } e_1 \text{ else } e_2 \mid \text{E } e \mid v \text{ E} \mid \text{fold } E \mid \text{unfold } E
\end{gathered}$$

$$\textsc{E-IfThenElseTrue}\ \frac{}{\text{if true then } e_1 \text{ else } e_2 \mapsto e_1} \qquad \textsc{E-IfThenElseFalse}\ \frac{}{\text{if false then } e_1 \text{ else } e_2 \mapsto e_2} \qquad \textsc{E-App}\ \frac{}{(\lambda x : \tau.\ e)\ v \mapsto e[v/x]}$$

$$\textsc{E-Step}\ \frac{e \mapsto e'}{E[e] \mapsto E[e']} \qquad \textsc{E-Fold}\ \frac{}{\text{unfold}(\text{fold } v) \mapsto v}$$

Figure 3: Call-by-value small step semantics of STLC + recursive types with evaluation context.

$$\boxed{\Gamma \vdash e : \tau}$$

$$\text{T-Unit}\ \frac{}{\Gamma \vdash \langle\rangle : 1} \qquad \text{T-True}\ \frac{}{\Gamma \vdash \text{true} : \text{bool}} \qquad \text{T-False}\ \frac{}{\Gamma \vdash \text{false} : \text{bool}} \qquad \text{T-Var}\ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$$

$$\text{T-IfThenElse}\ \frac{\Gamma \vdash e : \text{bool} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau} \qquad \text{T-Abs}\ \frac{x : \tau_1, \Gamma \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2} \qquad \text{T-App}\ \frac{\Gamma \vdash e_1 : \tau_2 \to \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau_1}$$

$$\text{T-Fold}\ \frac{\Gamma \vdash e : \tau[\mu\alpha.\tau/\alpha]}{\Gamma \vdash \text{fold } e : \mu\alpha.\tau} \qquad \text{T-Unfold}\ \frac{\Gamma \vdash e : \mu\alpha.\tau}{\Gamma \vdash \text{unfold } e : \tau[\mu\alpha.\tau/\alpha]}$$

Figure 4: Typing judgments of STLC + recursive types.

Let us see how the recursive type allows to type the term Ω. We first define the term $\text{SA} \triangleq \lambda x : ?.\ xx$. As such, $\Omega = \text{SA SA}$. We proceed in three steps: type x, then type SA and finally type Ω.

We fix a temporary type τ, and say that $x : \mu\alpha.\ \alpha \to \tau$. By unfolding this type once, we get $(\mu\alpha.\ \alpha \to \tau) \to \tau$, which would take an argument that has the type of x. As such, we can *encode*[3] the self-application with the function $\lambda x : \mu\alpha.\ \alpha \to \tau.\ (\text{unfold } x)\ x$. The self-application function then takes an argument of type $\mu\alpha.\ \alpha \to \tau$ and returns a value of type τ, i.e. $\cdot \vdash \text{SA} : (\mu\alpha.\ \alpha \to \tau) \to \tau$. Finally, if we encode $\Omega \triangleq \text{SA (fold SA)}$, it is then the application of an argument of type $(\mu\alpha.\ \alpha \to \tau)$ to a function of type $(\mu\alpha.\ \alpha \to \tau) \to \tau$, i.e. $\cdot \vdash \Omega : \tau$. It means in particular that Ω can be typed with any type τ. Intuitively, we can think of it as an expression that never terminate behaves similarly to any type, because we will never be able to tell the difference between one type or another.

4.3 Step-indexed logical relations for type safety of STLC with recursive types

To prove type safety for the STLC extended with recursive types, we might be tempted to try to extend the $\mathcal{V}[\![-]\!]$ logical relation for the recursive type case:

$$\mathcal{V}[\![\mu\alpha.\tau]\!] \quad \triangleq \quad \{\text{fold } v \mid \text{unfold (fold } v) \in \mathcal{E}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\}$$

but notice that unfold (fold v) $\mapsto v$, so simplifying this definition, and noticing that v is a value, we get:

$$\mathcal{V}[\![\mu\alpha.\tau]\!] \quad \triangleq \quad \{\text{fold } v \mid v \in \mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\}$$

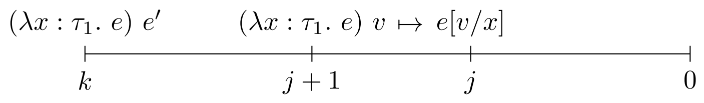
But now $\mathcal{V}[\![\tau]\!]$ is no longer well-founded, because the definition of $\mathcal{V}[\![\mu\alpha.\tau]\!]$ refers to $\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]$, which is indexed by a larger type. To address this problem, we will index our logical relations with a step index k, in addition to a type. The idea is that we will define new relations, $\mathcal{V}_k[\![\tau]\!]$ and $\mathcal{E}_k[\![\tau]\!]$, that contain values and expressions that behave like type τ *up to k reduction steps.*

$$\begin{aligned}
\mathcal{V}_k[\![\text{bool}]\!] &\triangleq \{\text{true}, \text{false}\} \\
\mathcal{V}_k[\![\tau_1 \to \tau_2]\!] &\triangleq \{\lambda x : \tau_1.\ e \mid \forall j < k.\ \forall v \in \mathcal{V}_j[\![\tau_1]\!].\ e[v/x] \in \mathcal{E}_j[\![\tau_2]\!]\} \\
\mathcal{V}_k[\![\mu\alpha.\tau]\!] &\triangleq \{\text{fold } v \mid \forall j < k.\ v \in \mathcal{V}_j[\![\tau[\mu\alpha.\tau/\alpha]]\!]\} \\
\mathcal{E}_k[\![\tau]\!] &\triangleq \{e \mid \forall j < k.\ \forall e'.\ e \mapsto_j e' \land \text{irred}(e') \Rightarrow e' \in \mathcal{V}_{k-j}[\![\tau]\!]\} \\
\mathcal{G}_k[\![\cdot]\!] &\triangleq \emptyset \\
\mathcal{G}_k[\![\Gamma, x : \tau]\!] &\triangleq \{\gamma[x \mapsto v] \mid \gamma \in \mathcal{G}_k[\![\Gamma]\!] \land v \in \mathcal{V}_k[\![\tau]\!]\}
\end{aligned}$$

We remark that we do not define the logical relation for the type α. The reason is quite subtle: the type α exists only under the $\mu\alpha.\tau$, which is immediately replaced. More precisely, it is because the α only appears under the μ.

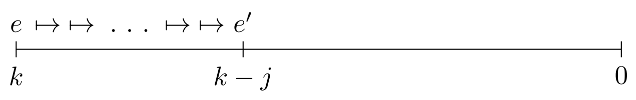
To understand the definition of $\mathcal{V}_k[\![\tau_1 \to \tau_2]\!]$, consider the following timeline:

[3] It is not an issue to slightly modify the function this way, because it behaves exactly the same as SA in the untyped lambda-calculus.


$(\lambda x : \tau_1.\ e)\ e'$
$(\lambda x : \tau_1.\ e)\ v \mapsto e[v/x]$
k
$j+1$
j
0

At the beginning, we have k reduction steps left to take. At some point in the future, after the program runs for some number of steps, say when we have $j+1$ steps left (where $j < k$ by definition), the lambda expression $\lambda x : \tau_1.\ e$ receives a value as an argument, and we take one reduction step to reduce the application $(\lambda x : \tau_1.\ e)v$ to $e[v/x]$, so we are left with j reduction steps to take, during which it has to behaves like a value of type τ_2.

The definition of $\mathcal{E}_k[\![\tau]\!]$ uses a different timeline:



At the beginning, we have k reduction steps left to reduce e. We take j (for some $j < k$) reduction steps and end up reducing e to e', and at that point, we have $k - j$ reduction steps left.

The fundamental property has the same form than the previous example, namely $\Gamma \vdash e : \tau \Rightarrow \Gamma \vDash e : \tau$. However, the definition of semantic type safety has changed:

$$\Gamma \vDash e : \tau \triangleq \forall k \geq 0. \forall \gamma \in \mathcal{G}_k[\![\Gamma]\!]. \gamma(e) \in \mathcal{E}_k[\![\tau]\!]$$

Before looking more closely to the proof of the fundamental theorem, we introduce a helpful small lemma

Lemma (Monotonicity)**.** *If $v \in \mathcal{V}_k[\![\tau]\!]$ and $j \leq k$ then $v \in \mathcal{V}_j[\![\tau]\!]$.*

Proof. Induction on the structure of type τ. The reader is encouraged to try proving the lemma for the $\tau_1 \to \tau_2$ case. It may also be illustrative to try proving the lemma with $j = k - 1$ and observe where the proof fails. ☐

Roughly, the monotonicity lemma4.3 states that if an expression behaves like a certain type for k steps, then it also behaves like this same type for any less steps.

Let look closer to the fundamental property proof for the step-indexed logical relation.

Theorem (Fundamental property)**.** *If $\Gamma \vdash e : \tau$, then $\Gamma \vDash e : \tau$*

Proof. We proceed by induction on the typing judgment. We will focus only on the case T-Fold.

- Case T-Fold $\dfrac{\Delta; \Gamma \vdash e : \tau[\mu\alpha.\tau/\alpha]}{\Delta; \Gamma \vdash \text{fold } e : \mu\alpha.\tau}$: We want to show that $\Gamma \vDash \text{fold } e : \tau$.

Suppose $k \geq 0$ and $\gamma \in \mathcal{G}_k[\![\Gamma]\!]$. We want to show that

$$\gamma(\text{fold } e) \in \mathcal{E}_k[\![\mu\alpha.\tau]\!]$$

Note that we can push the substitution into the fold, so this is equivalent to $\text{fold}(\gamma(e)) \in \mathcal{E}_k[\![\mu\alpha.\tau]\!]$.

Suppose $j < k$ and $\text{fold}(\gamma(e)) \mapsto^j e' \wedge \text{irred}(e')$. We want to show that

$$e' \in \mathcal{V}_{k-j}[\![\mu\alpha.\tau]\!]$$

Because the only way to reduce a (fold e) expression (according to the operational semantic) is to reduce the expression e inside the fold, we know that it exists a certain amount of steps j_1 such that $\text{fold}(\gamma(e)) \mapsto^{j_1} \text{fold}(e_1)$, where $j_1 \leq j$ and $\text{irred}(e_1)$. We instantiate the inductive hypothesis, $\Gamma \vDash e : \tau[\mu\alpha.\tau/\alpha]$ with γ and k (note that $\gamma \in \mathcal{G}_k[\![\Gamma]\!]$), which gives us

$$\gamma(e) \in \mathcal{E}_k[\![\tau[\mu\alpha.\tau/\alpha]]\!]$$

by the definition of $\vDash$. We instantiate $\mathcal{E}_k[\![\tau[\mu\alpha.\tau/\alpha]]\!]$ with j_1 and e_1, noting that $\text{irred}(e_1)$ and since $j_1 \leq j < k$, we also have $j_1 < k$.

Therefore, by unfolding the definition of $\mathcal{E}_k[\![-]\!]$ we know that

$$e_1 \in \mathcal{V}_{k-j}[\![\tau[\mu\alpha.\tau/\alpha]]\!]$$

However, because e_1 is irreducible in the value relation, then we know that it is a value $e_1 = v_1$. Moreover, $\text{fold}(v_1)$ is also a value. If we sum up our hypotheses at this point, we know that $\text{fold}(e) \mapsto^j e'$ and $\text{fold}(e) \mapsto^{j_1} \text{fold}(v_1)$, where both e' and $\mathrm{fold}(v_1)$ are irreducible. Therefore, $\mathrm{fold}(v_1) = e'$ and $j_1 = j$. Looking back to the last goal, it then suffices to show that

$$\mathrm{fold}(v_1) \in \mathcal{V}_{k-j_1}[\![\mu\alpha.\tau]\!]$$

Suppose $j' < k - j$. We must show that

$$v_1 \in \mathcal{V}_{j'}[\![\tau[\mu\alpha.\tau/\alpha]]\!]$$

This follows from the monotonicity lemma.

□

4.4 Generalising step-indexing to cover mutable state

The technique of step-indexing turns out to be very useful for dealing with dependencies resulting in non-wellfounded definitions. For recursive types, it side steps the issue of ever-growing types that rise out of unfolding. It can also help with other kinds of dependencies, for example the cyclic dependencies that can arise in a language with mutable state.

In order to extend step-index to allow for mutable state, we need to not only keep track of the best-before date of terms, as well as the current state of the heap in the indexing. In the same way that the preceding logical relation refers to lower natural numbers, we would also need to talk about the possible future heap states reachable from the current heap state. This idea of reachability is encapsulated by *Kripke frames* which consist of a set of worlds $\mathbb{W}$ and a reachability relation $w \sqsubseteq w'$ between worlds. The set of worlds is then used to index the logical relation, leading to the concept of *Kripke logical relations.*

Phrasing the previous logical relation in these terms, the set of worlds is $\mathbb{N}$ and the reachability relation is $<$. In the case of mutable state, the worlds would include the state of the heap and the reachable worlds would be given by heaps which you could reach by new allocations or modifying the heap. For more information, we refer the reader to Amal Ahmed's dissertation [Ahm04, Chapters 2-3]

5 Polymorphism

In this last section, we give a flavor about defining a logical relation for parametric polymorphism.

5.1 Formalization of System F

$$\begin{gathered}
\tau ::= \ \mathrm{bool} \mid \tau \to \tau \mid \alpha \mid \forall\alpha.\tau \\
e ::= x \mid \mathrm{true} \mid \mathrm{false} \mid \lambda x.\ e \mid \Lambda\alpha.\ e \mid e[\tau] \mid e\ e \\
v ::= \mathrm{true} \mid \mathrm{false} \mid \lambda x : \tau.\ e \mid \Lambda\alpha.v \\
E ::= [] \mid \text{if E then } e_1 \text{ else } e_2 \mid \mathrm{E}\ e \mid v\ \mathrm{E} \mid \mathrm{E}[\tau]
\end{gathered}$$

$$\textsc{E-IfThenElseTrue}\quad \frac{}{\text{if true then } e_1 \text{ else } e_2 \mapsto e_1} \qquad \textsc{E-IfThenElseFalse}\quad \frac{}{\text{if false then } e_1 \text{ else } e_2 \mapsto e_2} \qquad \textsc{E-App}\quad \frac{}{(\lambda x : \tau.\ e)\ v \mapsto e[v/x]}$$

$$\textsc{E-Step}\quad \frac{e \mapsto e'}{E[e] \mapsto E[e']} \qquad \textsc{E-TApp}\quad \frac{}{(\Lambda\alpha.e)[\tau] \mapsto e[\tau/\alpha]}$$

Figure 5: Call-by-value small step semantics of System F with evaluation context. $(\Lambda\alpha.e)[\tau] \mapsto e[\tau/\alpha]$.

$$\boxed{\Delta; \Gamma \vdash e : \tau}$$

$$\textsc{T-True}\quad \frac{}{\Delta; \Gamma \vdash \text{true} : \text{bool}} \qquad \textsc{T-False}\quad \frac{}{\Delta; \Gamma \vdash \text{false} : \text{bool}} \qquad \textsc{T-Var}\quad \frac{\Gamma(x) = \tau}{\Delta; \Gamma \vdash x : \tau}$$

$$\textsc{T-IfThenElse}\quad \frac{\Delta; \Gamma \vdash e : \text{bool} \qquad \Delta; \Gamma \vdash e_1 : \tau \qquad \Delta; \Gamma \vdash e_2 : \tau}{\Delta; \Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau} \qquad \textsc{T-Abs}\quad \frac{x : \tau_1, \Delta; \Gamma \vdash e : \tau_2}{\Delta; \Gamma \vdash \lambda x : \tau_1.\ e : \tau_1 \to \tau_2}$$

$$\textsc{T-App}\quad \frac{\Delta; \Gamma \vdash e_1 : \tau_2 \to \tau_1 \qquad \Delta; \Gamma \vdash e_2 : \tau_2}{\Delta; \Gamma \vdash e_1\ e_2 : \tau_1} \qquad \textsc{T-TAbs}\quad \frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau}$$

$$\textsc{T-Tapp}\quad \frac{\Delta; \Gamma \vdash e : \forall\alpha.\tau \qquad \Delta \vdash \tau'}{\Delta; \Gamma \vdash e[\tau'] : \tau[\tau'/\alpha]}$$

Figure 6: Typing judgments of System F. Δ is a *type environment* consisting of type variables α.

5.2 Motivation and free theorems

The most well-known example of a free theorem concerns terms $\cdot \vdash e : \Lambda\alpha.\alpha \to \alpha$. In a language supporting parametric polymorphism such as System F, we know that e must behave like the polymorphic identity function. By this, we mean that for any closed type τ and and term $\cdot \vdash e' : \tau$, if $e\ [\tau]\ e'$ terminates then we know that it will reduce to e'. Intuitively, in a language with parametric polymorphism we are unable to inspect the type of terms, so we must produce something of type α in a uniform way. In a similar vein, parametric polymorphism also tells us that any term $\cdot \vdash e : \Lambda\alpha.\alpha \to \tau$ will behave like a constant function or that any term $\cdot \vdash e : \Lambda\alpha.\alpha$ must not terminate.

5.3 Logical relation for parametricity of System F

In this section, we define step-by-step the logical relation for polymorphism and explain what misses in a naive approach, and how to solve the problem.

We define a binary logical relation, $(v_1, v_2) \in \mathcal{V}[\![\tau]\!]$, saying that the values v_1 and v_2 are related under the type τ.

We first start by lifting the unary logical relation of type safety of STLC to a binary logical relation, which states how to values of the same type behaves similarly. For instance, two booleans behave similarly if they are equals; and two functions are related if given related inputs, returns related outputs. Formally,

$$\begin{aligned}
\mathcal{V}[\![\text{bool}]\!] &\triangleq \{(\text{true},\ \text{true}),\ (\text{false},\ \text{false})\} \\
\mathcal{V}[\![\tau_1 \to \tau_2]\!] &\triangleq \{(\lambda x : \tau_1.\ e_1,\ \lambda x : \tau_1.\ e_2) \mid \forall (v_1,\ v_2) \in \mathcal{V}[\![\tau_1]\!].\ (e_1[v_1/x],\ e_2[v_2/x]) \in \mathcal{E}[\![\tau_2]\!]\}
\end{aligned}$$

Then, let us try to extend the definition with the polymorphic case. Intuitively, what is expected is that, two values of type $\forall\alpha.\ \tau$ needs to be of the form $\Lambda\alpha.e$, and they are related if giving them a type τ to instantiate α, they should be related in the expression relation.

$$\mathcal{V}[\![\forall\alpha.\ \tau]\!] \triangleq \{(\Lambda\alpha.e_1,\ \Lambda\alpha.e_2) \mid \forall(\tau_1,\ \tau_2).\ (e_1[\tau_1/\alpha],\ e_2[\tau_2/\alpha])) \in \mathcal{E}[\![\tau[?/\alpha]]\!]\}$$

However, both values might receive a different type, and then it is not clear with which type we are supposed to substitute α in the expression relation. Actually, there is no good answer: it cannot be either τ_1 or τ_2. The solution is to keep τ unchanged, and keep track of the type substitutions done all over the execution, recorded into a structure $\rho = \{\alpha \mapsto (\tau_1, \tau_2), \ldots\}$. We parameterize the logical relation by the type substitution ρ.

$$\begin{aligned}
\mathcal{V}[\![\text{bool}]\!]_\rho &\triangleq \{(\text{true},\ \text{true}),\ (\text{false},\ \text{false})\} \\
\mathcal{V}[\![\tau_1 \to \tau_2]\!]_\rho &\triangleq \{(\lambda x : \tau_1.\ e_1,\ \lambda x : \tau_1.\ e_2) \mid \forall (v_1,\ v_2) \in \mathcal{V}[\![\tau_1]\!]_\rho.\ (e_1[v_1/x],\ e_2[v_2/x]) \in \mathcal{E}[\![\tau_2]\!]_\rho\} \\
\mathcal{V}[\![\forall\alpha.\ \tau]\!]_\rho &\triangleq \{(\Lambda\alpha.e_1,\ \Lambda\alpha.e_2) \mid \forall(\tau_1,\ \tau_2).\ (e_1[\tau_1/\alpha],\ e_2[\tau_2/\alpha])) \in \mathcal{E}[\![\tau[\tau/\alpha]]\!]_{\rho[\alpha \mapsto (\tau_1, \tau_2)]}\}
\end{aligned}$$

It now remains to define the α case. Let us give it a shot.

$$\mathcal{V}[\![\alpha]\!]_\rho \triangleq \{(v_1,\ v_2) \mid \rho(\alpha) = (\tau_1, \tau_2) \wedge \ldots\}$$

And at this point, we do not know how to relate the values. This is actually where the polymorphism shines and reveal it's full power. The relation between the values of two type can actually be chosen by the user: the true power of parametric polymorphism is the choice of this relation.

For each type variable, we get to choose a relation $R \in \mathcal{R}[\tau_1, \tau_2]$ that relates values v_1 of type τ_1 with values v_2 of type τ_2. This relation is also recorded in the substitution $\rho = \{\alpha \mapsto (\tau_1, \tau_2, R), \ldots\}$. It finally gives us a way to relates the values of type α.

The final definition of the binary logical relation is the following:

$$\begin{array}{rcl}
\mathcal{V}[\![\text{bool}]\!]_\rho & \triangleq & \{(\text{true},\ \text{true}),\ (\text{false},\ \text{false})\} \\
\mathcal{V}[\![\tau_1 \to \tau_2]\!]_\rho & \triangleq & \{(\lambda x : \tau_1.\ e_1,\ \lambda x : \tau_1.\ e_2) \mid \forall (v_1,\ v_2) \in \mathcal{V}[\![\tau_1]\!]_\rho.\ (e_1[v_1/x],\ e_2[v_2/x]) \in \mathcal{E}[\![\tau_2]\!]_\rho\} \\
\mathcal{V}[\![\forall \alpha.\ \tau]\!]_\rho & \triangleq & \{(\Lambda\alpha.e_1,\ \Lambda\alpha.e_2) \mid \forall (\tau_1,\ \tau_2).\ R \in \mathcal{R}[\tau_1, \tau_2].\ (e_1[\tau_1/\alpha],\ e_2[\tau_2/\alpha])) \in \mathcal{E}[\![\tau[\tau/\alpha]]\!]_{\rho[\alpha \mapsto (\tau_1, \tau_2, R)]}\} \\
\mathcal{V}[\![\alpha]\!]_\rho & \triangleq & \{(v_1,\ v_2) \mid \rho(\alpha) = (\tau_1, \tau_2, R) \wedge (v_1,\ v_2) \in R\}
\end{array}$$

We didn't define the expression relation, nor the fundamental theorem during the lecture. We refer the interested reader to the Logical Relation course from OPLSS 2015, Lectures 3, 4, and 5. The videos of the lectures are available online [Ahm15].

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