Introduction to Proof Theory

Lecture 3 Sequent calculus

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Outline

NJ soundness and completeness

Curry-Howard correspondence

Sequent calculus LJ

LJ soundness and completeness

A derivation of a formula A is a tree of formulas in which every formula that is not an <u>assumption</u> is the conclusion of a rule application.

The assumptions in the deduction that are not discharged by any rule in it are the open assumptions of the deduction.

If every assumption is discharged, we have a proof of A.

Completeness of NJ



Completeness of NJ

[<u>]</u> _ E • PL9. : elim a.: 1 **7**τ $L \rightarrow B$ Reptangthen Rip + B→A Slip + B R of length < r Inductive step ß B-JA MP Α NJHBJA & NJHB ----- A В-A

Soundness of NJ



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Some history

λ -calculus with product and sum

$$L, M, N := 2 | \lambda_{x}, M | M N$$

$$| \langle M, N \rangle | \pi_{x}(M) | \pi_{y}(M)$$

$$| in_{x}(M) | in_{y}(M) | case(L, x \rightarrow M, y \rightarrow N) | \mathcal{E}(M)$$

$$[x:A]$$

$$\vdots$$

$$\frac{M:B}{\lambda_{x}, M : A \rightarrow B} \rightarrow_{I} \frac{M.A \rightarrow B}{(MN): B} \xrightarrow{N:A} \longrightarrow_{E}$$



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LJ soundness and completeness

From natural deduction to sequent calculus

hentren 1935



Rules of LJ ٦ =) ($\underbrace{A, [-] c & B, [-] c}_{A \lor B, [, ['-] c}$ レット $\operatorname{init} \frac{}{p, \Gamma \Rightarrow p}$ $^{\perp}\overline{\perp,\Gamma\Rightarrow C}$ initial sequet ${}_{\wedge_{\mathsf{R}}}\frac{\Gamma\Rightarrow A\quad \Gamma\Rightarrow B}{\Gamma\Rightarrow A\wedge B}$ $\frac{A, B, \Gamma \Rightarrow C}{-A \land B, \Gamma \Rightarrow C}$ $\nabla_{\mathsf{R}} \frac{\Gamma \Rightarrow A_{i}}{\Gamma \Rightarrow A_{0} \lor A_{1}} i \in \{0, 1\} \quad \underbrace{\overbrace{\Gamma \Rightarrow A_{0}}^{i} f_{i}}_{\Gamma \Rightarrow A_{0} \lor A_{1}} i \in \{0, 1\}$ $_{\vee_{L}}\frac{A,\Gamma\Rightarrow C\quad B,\Gamma\Rightarrow C}{A\vee B,\Gamma\Rightarrow C}$ $\overbrace{\rightarrow_{\mathsf{R}}}^{\mathsf{A},\,\Gamma\Rightarrow B} \overbrace{\Gamma\Rightarrow A\to B}^{\mathsf{A},\,\Gamma\Rightarrow} A \xrightarrow{\mathsf{F}} B$ $\rightarrow_{\scriptscriptstyle \mathsf{L}} \frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}$ HA-B $\operatorname{cut} \frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}$ $\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C} = A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}$

Derivations in LJ, formally

1.1 => 1 1, Γ => 1 P1 Ρ, T FNA A C T =) A has a derivation

A derivation in LJ is a (rooted) tree, whose nodes are labelled by sequents, and such that:

- The leaves of the tree initial sequent:
- The sequents occupying intermediate nodes in the tree are obtained from the sequents occupying the nodes directly above them by means of a correct application of an inference rule:
- The root of the tree is the conclusion of the derivation (the endsequent).

We denote by $\vdash_{I,I} \Gamma \Rightarrow C$ derivability of $\vdash_{I,I} \Gamma \Rightarrow C$ in LJ. We say that $\Gamma \Rightarrow C$ is derivable if there is a LJ derivation for it.

Example $\frac{A,\Gamma=>B}{\Gamma=>A\to B}$ Derivation of =) fr -> (q -> (p = q)) F⇒B (=) A NR I =) AAB init h,q=)q ∧r れ,9=>れ $\frac{\mu, q = \psi^{A}q}{\mu = \varphi^{A}(\psi^{A}q)} \rightarrow R$ $= \psi^{A}(q) \rightarrow (\psi^{A}q) \rightarrow R$ -) R

The rules for implication Hip C => + => C (=) Ai (=) A vA $\xrightarrow{\rightarrow_{L}} \underbrace{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}_{A \to B, \Gamma \Rightarrow C}$ $\xrightarrow{\to_{\mathsf{R}}} \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$ init 8,11 =) 小 $(\hat{B}, \hat{\mu}) = (\hat{\mu} \rightarrow L)^{\vee R}$ $(\hat{\mu} \rightarrow L)^{\vee R}$ $(\hat{\mu} \rightarrow L)^{\vee R}$ 8,1 => 上 B=) かっ」 B =) μ v (μ->⊥) vr <u>⊥=>⊥</u> <u>⊥</u>=>⊥ $\frac{(\bigwedge \vee (\bigwedge \rightarrow \bot)) \rightarrow \bot \Rightarrow \bot}{\Rightarrow ((\bigwedge \vee (\bigwedge \rightarrow \bot)) \rightarrow \bot) \rightarrow \bot} \rightarrow_{R}$ $B = (hv(h \rightarrow \perp)) \rightarrow \perp$ --- (hv-h)

The cut rule

