Introduction to Proof Theory

Lecture 3
Sequent calculus

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Outline

NJ soundness and completeness

Curry-Howard correspondence

Sequent calculus LJ

LJ soundness and completeness

Deduction trees

A derivation of a formula A is a tree of formulas in which every formula that is not an assumption is the conclusion of a rule application.

The assumptions in the deduction that are not discharged by any rule in it are the open assumptions of the deduction.

If every assumption is discharged, we have a proof of A.

Completeness of NJ

If $\vdash_{NJ} A$ then $\vdash_{\mathcal{H}ip} A$

By induction on the length of the $\mathcal{H}ip$ derivation

<u>Base case</u>: A is an instance of an $\mathcal{H}ip$ axiom

- PL1.

$a_1 : B$

$a_2 : A$

$$\dfrac{\dfrac{[B]}{A \to B}\ \to_I}{B \to (A \to B)}\ \to_I$$

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\ \to_I$$

Completeness of NJ

- PL9. :

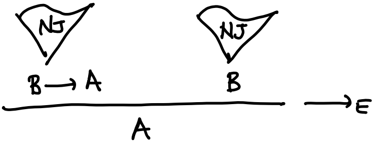
elim $\left\{ \dfrac{[\bot]}{B} \right. \bot E$ $\qquad a_1 : \bot$

$\left\{ \dfrac{B}{\bot \to B} \right. \to I$

Inductive step: $\mathcal{H}_{ip} \vdash A$ of length n $\qquad \mathcal{H}_{ip} \vdash B \to A \qquad \mathcal{H}_{ip} \vdash B$ of length $< n$

$$\dfrac{B \to A \qquad B}{A}\ mp$$

$NJ \vdash B \to A$ & $NJ \vdash B$

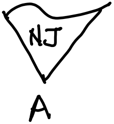


Soundness of NJ

If $\vdash_{NJ} A$ then $\vdash_{\mathcal{X}ip} A$

If $\Gamma \vdash_{NJ} A$ then $\vdash_{\mathcal{X}ip} C \to A$, with $C = \bigwedge \Gamma$

$\Gamma \vdash_{\mathcal{X}ip} A$



$\text{size}(B) = 1$

$\text{size}\left(\dfrac{\pi_1 \; \ldots \; \pi_n}{B}\right) = 1 + \sum_i \text{size}(\pi_i)$

Base case: A need $\vdash_{\mathcal{X}ip} A \to A$ ✓

Ind. step:

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}$$

(IH) $A \vdash_{NJ} B \implies A \vdash_{\mathcal{X}ip} B$

WTP: $\vdash_{NJ} A \to B \implies \vdash_{\mathcal{X}ip} A \to B$

Soundness of NJ

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Some history

λ-calculus with product and sum

Proofs as λ-terms

Proofs as λ-terms

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From natural deduction to sequent calculus

Rules of **LJ**

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow p} \qquad \bot\ \frac{}{\bot, \Gamma \Rightarrow C}$$

$$\wedge_L\ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \wedge_R\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad \vee_R\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ i \in \{0, 1\}$$

$$\rightarrow_L\ \frac{A \rightarrow B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \rightarrow B, \Gamma \Rightarrow C} \qquad \rightarrow_R\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$\text{cut}\ \frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}$$

Derivations in **LJ**, formally

A derivation in **LJ** is a (rooted) tree, whose nodes are labelled by sequents, and such that:

- The leaves of the tree initial sequent:
- The sequents occupying intermediate nodes in the tree are obtained from the sequents occupying the nodes directly above them by means of a correct application of an inference rule;
- The root of the tree is the conclusion of the derivation (the endsequent).

We denote by $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow C$ derivability of $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow C$ in **LJ**. We say that $\Gamma \Rightarrow C$ is derivable if there is a **LJ** derivation for it.

Example

The rules for implication

$$\to_L \frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad \to_R \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

The cut rule

$$\text{cut}\,\frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}$$

Other structural rules

A rule is admissible in **LJ** if, whenever the premiss(es) of the rule are derivable, then the conclusion of the rule is derivable.

$$\text{wk}\,\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C} \qquad \text{ctr}\,\frac{A, \Gamma \Rightarrow C}{A, A, \Gamma \Rightarrow C}$$

Outline

Soundness of **LJ**

Completeness of **LJ**