Introduction to Proof Theory

Lecture 5

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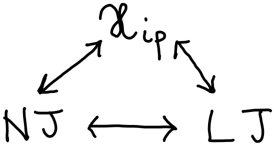
Outline

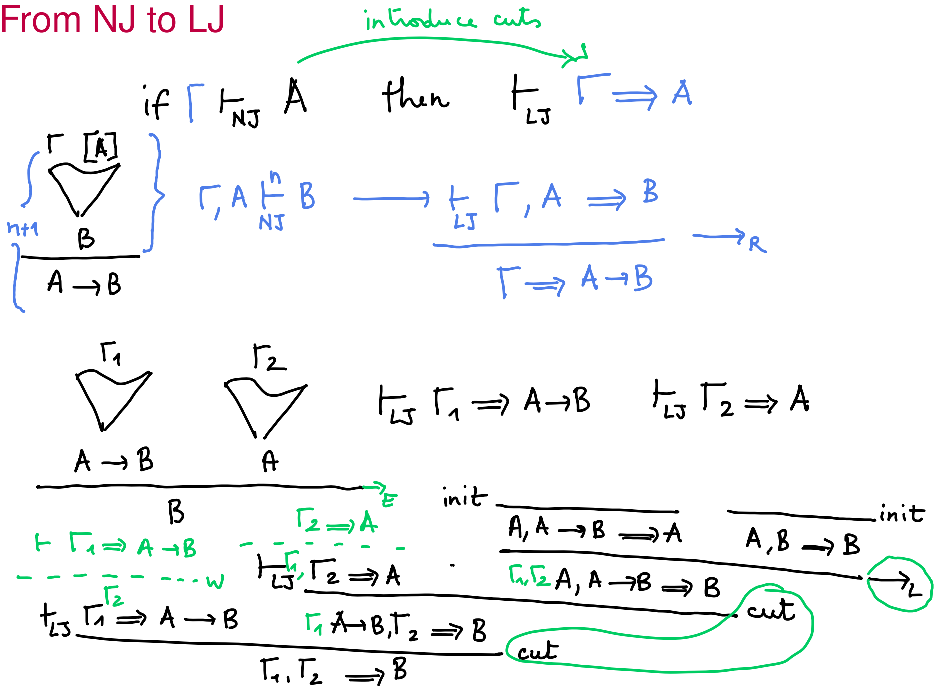


From LJ to NJ and back

Consistency, cut elimination and *PA*

Conclusions

From NJ to LJ



introduce cuts
if Γ ⊢NJ A then ⊢LJ Γ ⟹ A
Γ [A]
n+1
B
A → B
Γ, A ⊢ⁿNJ B
⊢LJ Γ, A ⟹ B
Γ ⟹ A → B
→R
Γ1
Γ2
A → B
A
B
→E
⊢LJ Γ1 ⟹ A → B
⊢LJ Γ2 ⟹ A
init
A, A → B ⟹ A
A, B ⟹ B
Γ1, Γ2 A, A → B ⟹ B
→L
Γ2 ⟹ A
⊢ Γ1 ⟹ A → B
W
⊢LJ Γ1, Γ2 ⟹ A
⊢LJ Γ2, Γ1 ⟹ A → B
Γ1 A → B, Γ2 ⟹ B
cut
Γ1, Γ2 ⟹ B

From NJ to LJ

From LJ to NJ

From LJ to NJ

Back to normalisation

Back to normalisation

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Consistency and cut elimination

Sequent calculus for classical propositional logic

Sequent calculus for classical first-order logic

$$\forall_L \frac{A(t), \forall x(A(x)), \Gamma \vdash \Delta}{\forall x(A(x)), \Gamma \vdash \Delta} \qquad \forall_R \frac{\Gamma \vdash \Delta, A(y)}{\Gamma \vdash \Delta, \forall x(A(x))}$$

$$\exists_L \frac{A(y), \Gamma \vdash \Delta}{\exists x(A(x)), \Gamma \vdash \Delta} \qquad \exists_R \frac{\Gamma \vdash \Delta, A(t), \exists x.A}{\Gamma \vdash \Delta, \exists x(A(x))}$$

$$\mathrm{eq}_1 \frac{x = y, \Gamma \Rightarrow \Delta, A(x), A(y)}{x = y, \Gamma \Rightarrow \Delta, A(x)} \qquad \mathrm{eq}_2 \frac{x = y, \Gamma \Rightarrow \Delta, A(y), A(x)}{x = y, \Gamma \Rightarrow \Delta, A(y)}$$

Condition: y does not occur free in Γ, Δ, A

PA, with sequents

Take the sequent calculus for first-order classical logic, and add:

- Initial sequents in correspondence to PA1. – PA6.

$$
\begin{array}{ll}
1 & \Rightarrow \forall x \neg(0 = s(x)) \\
2 & \Rightarrow \forall x \forall y (s(x) = s(y) \rightarrow x = y) \\
3 & \Rightarrow \forall x (x + 0 = x) \\
4 & \Rightarrow \forall x \forall y (x + s(y) = s(x + y)) \\
5 & \Rightarrow \forall x (x \cdot 0 = 0) \\
6 & \Rightarrow \forall x \forall y (x \cdot s(y) = (x \cdot y) + x)
\end{array}
$$

PA, with sequents

Take the sequent calculus for first-order classical logic, and add:

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5 & \Rightarrow \forall x (x \cdot 0 = 0) \\
6 & \Rightarrow \forall x \forall y (x \cdot s(y) = (x \cdot y) + x)
\end{array}
$$

- Induction rule corresponding to the induction axiom

$$
\text{ind}\ \frac{\Gamma \Rightarrow \Delta, A(0) \quad A(s(x)), \Gamma \Rightarrow \Delta, s(x)}{\Gamma \Rightarrow \Delta, \forall x (A(x))}
$$

Example

$$
\text{cut}\;\dfrac{\dfrac{}{\Rightarrow \mathrm{PA2.}} \qquad \text{∀}_L\dfrac{\text{∀}_L\dfrac{\to_L\dfrac{\text{init}\dfrac{}{sx = ssx \Rightarrow x = sx, sx = ssx} \qquad \text{init}\dfrac{}{x = sx, sx = ssx \Rightarrow x = sx}}{sx = ssx \to x = sx, sx = ssx \Rightarrow x = sx}}{\forall y(sx = sy \to x = y), sx = ssx \Rightarrow x = sx}}{\forall x \forall y(sx = sy \to x = y), sx = ssx \Rightarrow x = sx}}{\dfrac{sx = ssx \Rightarrow ssx = sx}{\to_R\dfrac{\to_L\;(x = sx) \to \bot, sx = ssx \Rightarrow \to \bot}{(x = sx) \to \bot \Rightarrow (sx = ssx) \to \bot}}}
$$

$$
\text{ind}\;\dfrac{\dfrac{\dfrac{}{\Rightarrow \mathrm{PA1.}} \qquad \forall_L\dfrac{\text{init}\dfrac{}{(0 = sx) \to \bot \Rightarrow (0 = s0) \to \bot}}{\forall x((0 = sx) \to \bot) \Rightarrow (0 = s0) \to \bot}}{\Rightarrow (0 = s(0)) \to \bot} \qquad \begin{matrix}\triangledown\\ C\end{matrix}}{\Rightarrow \forall x(x = s(x) \to \bot)}
$$

for $C := (x = sx) \to \bot \Rightarrow (sx = ssx) \to \bot$

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Perspectives