Introduction to Proof Theory

Lecture 5

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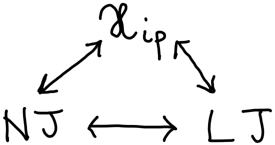
OPLSS 2023

Eugene, Oregon, June 26 - July 8, 2023

Outline



From LJ to NJ and back

Consistency, cut elimination and *PA*

Conclusions

From NJ to LJ

introduce cuts

if $\Gamma \vdash_{NJ} A$ then $\vdash_{LJ} \Gamma \Rightarrow A$

$$\left.\dfrac{\begin{matrix}\Gamma \quad [A] \\ \vdots \\ B\end{matrix}}{A \to B}\right\} n+1$$

$$\Gamma, A \vdash^{n}_{NJ} B \longrightarrow \vdash_{LJ} \Gamma, A \Rightarrow B$$

$$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \to_R$$

$$\dfrac{\begin{matrix}\Gamma_1 \\ \vdots \\ A \to B\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots \\ A\end{matrix}}{B} \to_E$$

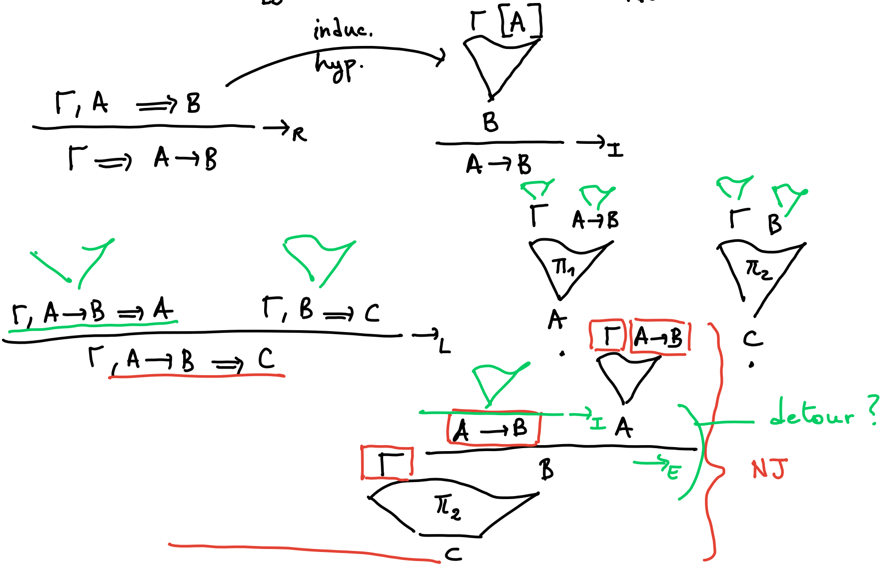
$$\vdash_{LJ} \Gamma_1 \Rightarrow A \to B \qquad \vdash_{LJ} \Gamma_2 \Rightarrow A$$

$$\dfrac{\dfrac{\vdash \Gamma_1 \Rightarrow A \to B}{\vdash_{LJ} \Gamma_1, \Gamma_2 \Rightarrow A \to B} W \qquad \dfrac{\dfrac{\Gamma_2 \Rightarrow A}{\vdash_{LJ} \Gamma_1, \Gamma_2 \Rightarrow A} \qquad \dfrac{\dfrac{}{A, A \to B \Rightarrow A}\,\text{init} \qquad \dfrac{}{A, B \Rightarrow B}\,\text{init}}{\Gamma_1, \Gamma_2, A, A \to B \Rightarrow B} \to_L}{\Gamma_1, A \to B, \Gamma_2 \Rightarrow B}\,\text{cut}}{\Gamma_1, \Gamma_2 \Rightarrow B}\,\text{cut}$$

From LJ to NJ

add detours

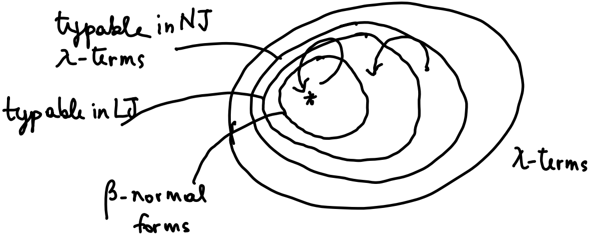
If $\vdash_{LJ} \Gamma \Rightarrow A$ then $\Gamma \vdash_{NJ} A$



Back to normalisation

no detours

$$\begin{tikzcd}
\vdash_{NJ} M : A \arrow[rr, "\text{normalisation}", "\beta\text{-reduction}"'] \arrow[d, bend right] \arrow[d, leftrightarrow] & & \text{normal } \vdash_{NJ} M' : A \arrow[d, bend left] \arrow[d, leftrightarrow, "\exists\text{ translation preserves normality}"'] \arrow[dll] \\
\vdash_{LJ} N : A \arrow[rr, "\text{cut elimination}"] & & \text{cut-free } \vdash_{LJ} A \arrow[ull]
\end{tikzcd}$$



if M normal

$$\vdash_{NJ} M : A \longleftrightarrow \vdash_{LJ} M : A$$

$$\updownarrow$$

cut free $\vdash_{LJ} M : A$

Back to normalisation

$\vdash_{NJ}$

$\nvdash_{LJ}$ $(\lambda x.\lambda y.x)(\lambda z.z)(\lambda z.z) = M$

$\vdash_{LJ}$ $\mathrm{translate}(M)$

Outline

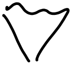
Consistency and cut elimination

$$\boxed{\vdash_{Hip}} \; A \quad \overset{\rightsquigarrow}{\text{compl.}} \quad \boxed{\vdash_{LJ\,+\,\text{cut}}} \Rightarrow A \quad \overset{\rightsquigarrow}{\text{cut elim.}} \quad \boxed{\vdash_{LJ} \Rightarrow A}$$

Hip is consistent?

$$\vdash_{Hip} \bot \;? \qquad \longrightarrow \qquad \vdash_{LJ} \Rightarrow \bot$$

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,\text{wk} \qquad \frac{\Gamma \Rightarrow A \qquad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}$$



$$\frac{\Rightarrow \bot}{}$$

Sequent calculus for classical propositional logic

intuitionistic $\quad \Gamma \Rightarrow C \quad$ fm $(\bigwedge \Gamma \to C)$ $\qquad \dfrac{\neg\neg A}{A}$

multiset of formulas

classical : $\quad \Gamma \Rightarrow \Delta \quad$ fm $(\bigwedge \Gamma \to \bigvee \Delta)$

Basic Proof Theory

G3cp

int

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\vee_R \qquad i \in \{0,1\}$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\to_R$$

$$\frac{\Gamma, A \to B \Rightarrow A, \Delta \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}\to_L$$

classical :

$$\frac{\Gamma \Rightarrow A_0, A_1, \Delta}{\Gamma \Rightarrow A_0 \vee A_1 \; \Delta}\vee_{R_c}$$

$$\dfrac{\dfrac{A \Rightarrow A, \bot}{\Rightarrow A, A \to \bot}\to_R}{\Rightarrow A \vee (A \to \bot)}\vee_R$$

$A \vee \neg A$

Sequent calculus for classical first-order logic

$$\frac{}{\Gamma \Rightarrow \Delta, x = x}\ \mathrm{eq}_0$$

$$\forall_L\ \frac{A(t), \forall x(A(x)), \Gamma \overset{\Rightarrow}{\vdash} \Delta}{\forall x(A(x)), \Gamma \vdash \Delta} \qquad \forall_R\ \frac{\Gamma \vdash \Delta, A(y)}{\Gamma \vdash \Delta, \forall x(A(x))}\ (*)$$

$$\exists_L\ \frac{A(y), \Gamma \vdash \Delta}{\exists x(A(x)), \Gamma \vdash \Delta}\ (*) \qquad \exists_R\ \frac{\Gamma \vdash \Delta, A(t), \exists x.A}{\Gamma \vdash \Delta, \exists x(A(x))}$$

$$\mathrm{eq}_1\ \frac{x = y, \Gamma \Rightarrow \Delta, A(x), A(y)}{x = y, \Gamma \Rightarrow \Delta, A(x)} \qquad \mathrm{eq}_2\ \frac{x = y, \Gamma \Rightarrow \Delta, A(y), A(x)}{x = y, \Gamma \Rightarrow \Delta, A(y)}$$

Condition: y does not occur free in Γ, Δ, A

$\forall x\ (x = x)$

$\forall x \forall y\ (x = y \rightarrow A(x) = A(y))$

PA, with sequents

Take the sequent calculus for first-order classical logic, and add:

- Initial sequents in correspondence to PA1. – PA6.

$$
\begin{array}{ll}
1 & \Rightarrow \forall x \neg (0 = s(x)) \\
2 & \Rightarrow \forall x \forall y (s(x) = s(y) \rightarrow x = y) \\
3 & \Rightarrow \forall x (x + 0 = x) \\
4 & \Rightarrow \forall x \forall y (x + s(y) = s(x + y)) \\
5 & \Rightarrow \forall x (x \cdot 0 = 0) \\
6 & \Rightarrow \forall x \forall y (x \cdot s(y) = (x \cdot y) + x)
\end{array}
$$

PA, with sequents

Take the sequent calculus for first-order classical logic, and add:

- Initial sequents in correspondence to PA1. – PA6.

$$
\begin{array}{ll}
1 & \Rightarrow \forall x \neg(0 = s(x)) \\
2 & \Rightarrow \forall x \forall y (s(x) = s(y) \rightarrow x = y) \\
3 & \Rightarrow \forall x (x + 0 = x) \\
4 & \Rightarrow \forall x \forall y (x + s(y) = s(x + y)) \\
5 & \Rightarrow \forall x (x \cdot 0 = 0) \\
6 & \Rightarrow \forall x \forall y (x \cdot s(y) = (x \cdot y) + x)
\end{array}
$$

- Induction rule corresponding to the induction axiom

$$
\text{ind}\ \frac{\Gamma \Rightarrow \Delta, A(0) \qquad A(x), \Gamma \Rightarrow \Delta, A(s(x))}{\Gamma \Rightarrow \Delta, \forall x(A(x))}
$$

Example

π :

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\Rightarrow \mathrm{PA2}.$}
\AxiomC{}
\RightLabel{}
\LeftLabel{init}
\UnaryInfC{$sx = ssx \Rightarrow x = sx, sx = ssx$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$x = sx, sx = ssx \Rightarrow x = sx$}
\LeftLabel{$\to_L$}
\BinaryInfC{$sx = ssx \to x = sx, sx = ssx \Rightarrow x = sx$}
\LeftLabel{$\forall_L$}
\UnaryInfC{$\forall y(sx = sy \to x = y), sx = ssx \Rightarrow x = sx$}
\LeftLabel{$\forall_L$}
\UnaryInfC{$\forall x \forall y(sx = sy \to x = y), sx = ssx \Rightarrow x = sx$}
\LeftLabel{cut}
\BinaryInfC{$sx = ssx \Rightarrow ssx = sx$}
\LeftLabel{$\to_L$}
\UnaryInfC{$(x = sx) \to \bot, sx = ssx \Rightarrow \to\bot$}
\LeftLabel{$\to_R$}
\UnaryInfC{$(x = sx) \to \bot \Rightarrow (sx = ssx) \to \bot$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\Rightarrow \mathrm{PA1}.$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$(0 = sx) \to \bot \Rightarrow (0 = s0) \to \bot$}
\LeftLabel{$\forall_L$}
\UnaryInfC{$\forall x((0 = sx) \to \bot) \Rightarrow (0 = s0) \to \bot$}
\BinaryInfC{$\Rightarrow (0 = s(0)) \to \bot$}
\AxiomC{π}
\noLine
\UnaryInfC{C}
\LeftLabel{ind}
\BinaryInfC{$\Rightarrow \forall x(x = s(x) \to \bot)$}
\end{prooftree}
$$

$\forall x\,(x \neq s(x))$

for $C := (x = sx) \to \bot \Rightarrow (sx = ssx) \to \bot$

Outline

Perspectives