Introduction to Barendregt's Lambda Cube

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Lecture 1

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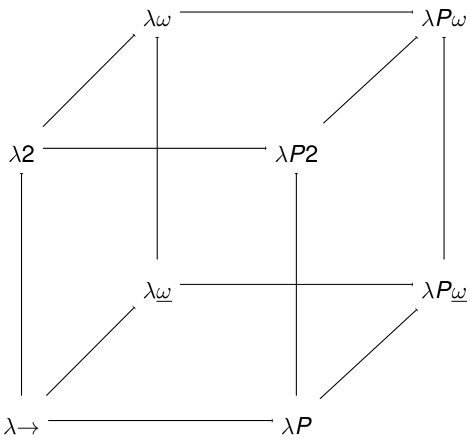

Figure: Lambda cube

- λ Untyped Lambda calculus
- $\lambda\!\rightarrow$ Simple types
- $\lambda 2$ Polymorphic types
- $\lambda\underline{\omega}$
- λP Dependent types
- Lambda cube
- Logic cube

Roadmap

Lambda Calculus

$\lambda \rightarrow$ - Simple types

Roadmap

Lambda Calculus

$\lambda \rightarrow$ - Simple types

λ-calculus, untyped lambda calculus

Decision Problem - Background

- Gottfried Wilhelm Leibniz - Characteristica universalis
- David Hilbert and Wilhelm Ackerman (1928)
  - **Entscheidungsproblem**, or Decision Problem:

    "Given all the axioms of math,
    is there an algorithm that can tell if a proposition is universally vali
    i.e. deducable from the axioms?"
- **Negative answers** (1935/36):
  - Alonzo Church - λ-calculus (equality)
  - Alan Turing - Turing Machines (halting problem)

  - Kurt Gödel - Incompleteness theorems (1931)

λ-calculus - 1930s

- Alonzo Church:
  - theory of functions - formalisation of mathematics (inconsistent)
  - successful model for computable functions - **λ-calculus**
  - **simply typed λ-calculus**
- Haskell Curry:
  - elimination of variables in logic - Moses Schönfinkel (1921)
  - successful model for computable functions - **Combinatory logic**
  - **Combinatory logic with types**
- Alan Turing :
  - formalisation of the concepts of algorithm and computation
  - **Turing Machines**

λ-calculus - expressiveness

- Expressiveness - Effective computability (mid 1930s)
  - **(Curry)** Equivalence of λ-calculus and Combinatory Logic
  - **(Kleene)** Equivalence of λ-calculus and recursive functions
  - **(Turing)** Equivalence of λ-calculus and Turing machines

Syntax

$$M \ ::= \ x \ | \ c \ | \ (MM) \ | (\lambda x.M)$$

x ranges over V, a countable set of variables
c ranges over C, a countable set of constants

Pure λ-calculus, if $C = \emptyset$

Conventions for minimizing the number of the parentheses:

- $M_1 M_2 M_3$ stands for $((M_1 M_2) M_3)$ application associates to left
- $\lambda x.y.M$ stands for $(\lambda x.(\lambda y.(M))$ abstraction associates to right
- $\lambda x.M_1 M_2 \ \equiv \ \lambda x.(M_1 M_2)$; application has priority over abstraction

Running example

$xyzx$	
$\lambda x.zx$	
$\mathbf{I} \equiv \lambda x.x$	combinator **I**
$\mathbf{K} \equiv \lambda xy.x$	combinator **K**
$\mathbf{S} \equiv \lambda xyz.xz(yz)$	combinator **S**
$\Delta \equiv \lambda x.xx$	selfapplication
$\mathbf{Y} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$	fixed point combinator
$\Omega \equiv \Delta\Delta \equiv (\lambda x.xx)(\lambda x.xx)$	higher-order function

Free and bound variables

Definition

(i) The set $FV(M)$ of free variables of M is defined inductively:

- $FV(x) = \{x\}$
- $FV(MN) = FV(M) \cup FV(N)$
- $FV(\lambda x.M) = FV(M) \setminus \{x\}$

(ii) A variable in M is bound if it is not free

- x is bound in M if it appears in a subterm of the form $\lambda x.N$

(ii) M is a closed λ-term (or *combinator*) if $FV(M) = \emptyset$
Λ^o denotes the set of closed λ-terms.

Example

- In $\lambda x.zx$, variable z is free, $FV(M) = \{z\}$
- Term $\lambda xy.xxy$ is closed, $FV(M) = \emptyset$

Reduction rules - operational semantics

α-reduction:

$$\lambda x.M \longrightarrow_\alpha \lambda y.M[x := y], \;\; y \notin FV(M)$$

β-reduction:

$$(\lambda x.M)N \longrightarrow_\beta M[x := N]$$

η-reduction:

$$\lambda x.(Mx) \longrightarrow_\eta M, \;\; x \notin FV(M)$$

α-conversion

Formalisation of the principal that the name of the bound variable is irrelevant

α-reduction:

$$\lambda x.M \longrightarrow_\alpha \lambda y.M[x := y], \quad y \notin FV(M)$$

In math, $f(x) = x^2 + 1$ and $f(y) = y^2 + 1$ same, $f(5) = 26$
$\lambda x.(x^2 + 1)$ and $\lambda y.(y^2 + 1)$ must be considered as equal

Proposition $\twoheadrightarrow_\alpha$ is an equivalence relation, notation $=_\alpha$

Proof.

Symmetry, the interesting case □

β-reduction

Formalisation of function evaluation

$$(\lambda x.M)N \longrightarrow_\beta M[x := N]$$

- $M[x := N]$ represents an evaluation of the function M with N being the value of the parameter x.
- $(\lambda x.M)N$ is a redex and $M[x := N]$ is a contractum
- β-conversion is the symmetric closure of $\twoheadrightarrow_\beta$ is an equivalence (with α-reduction), notation $\equiv_\beta$

- Barendregt's variable convention: If a term contains a free variable which would become bound after *beta*-reduction, that variable should be renamed.
- Renaming could be done also by using De Bruijn name free notation.

Example

$$(\lambda x.x^2 + 1)5 \longrightarrow_\beta 5^2 + 1 \rightarrow 26$$

η-conversion

Formalisation of extensionality

Definition

η-reduction:

$$\lambda x.(Mx) \longrightarrow_\eta M, \quad x \notin FV(M)$$

- This rule identifies two functions that always produce equal results if taking equal arguments.

Example

$$\lambda x.\mathbf{succ}x \longrightarrow_\eta \mathbf{succ}$$

$$(\lambda x.\mathbf{succ}x)2 \longrightarrow_\beta \mathbf{succ}2 \qquad \mathbf{succ}2$$

Properties

- confluence
- normal forms
- normalisation
- strong normalisation
- fixed point theorem
- expressiveness

Properties - Confluence

Theorem (Church-Rosser theorem)

If $M \longrightarrow\!\!\!\!\!\rightarrow N$ and $M \longrightarrow\!\!\!\!\!\rightarrow P$, then there exists S such that $N \longrightarrow\!\!\!\!\!\rightarrow S$ and $P \longrightarrow\!\!\!\!\!\rightarrow S$

The proof is deep and involved.

Corollary

- *If $M \longrightarrow\!\!\!\!\!\rightarrow N$ and $M \longrightarrow\!\!\!\!\!\rightarrow P$, then $N = P$*
- *The order of the applied reductions is arbitrary and always leads to the same result*
- *Reductions can be executed in parallel (parallel computing)*

Proof.

□

Normal forms

- $N \in \Lambda$ is a normal form (NF) if there is no S such that $N \longrightarrow S$
- $P \in \Lambda$ is normalising (has a normal form) if $P \twoheadrightarrow N$ and N is a normal form, then N is a NF of P
- $P \in \Lambda$ is strongly normalising (SN) if all reductions of P are finite

Notation: $\twoheadrightarrow$ will denote $\twoheadrightarrow_\beta \cup \longrightarrow_\alpha$

Theorem (uniqueness of NF)

Every lambda term has at most one normal form

Proof.

Exercise

□

Running example: β-normal forms

$xyzx$	normal form NF
$\mathbf{I} \equiv \lambda x.x$	normal form NF
$\mathbf{K} \equiv \lambda xy.x$	normal form NF
$\mathbf{S} \equiv \lambda xyz.xz(yz)$	normal form NF
$\mathbf{KI}(\mathbf{KII})$	strongly normalizing SN
$\Omega \equiv \Delta\Delta \equiv (\lambda x.xx)(\lambda x.xx)$	unsolvable
$\mathbf{KI}\Omega$	normalizing N
$\mathbf{Y} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$	head normalizing HN (solvable)

$\mathbf{KI}(\mathbf{KII}) \rightarrow \mathbf{KII} \rightarrow \mathbf{I}$
$\mathbf{KI}(\mathbf{KII}) \rightarrow \mathbf{I}$
$\Omega \rightarrow \Omega \rightarrow \Omega \rightarrow \Omega \rightarrow \dots$

Logic, conditionals, pairs

- Propositional logic in λ-calculus:

$$\top := \lambda xy.x \qquad \bot := \lambda xy.y \qquad \neg := \lambda x.x\bot\top$$
$$\wedge := \lambda xy.xy\bot \qquad \vee := \lambda xy.x\top y$$

Example

$$\top \vee A \longrightarrow (\lambda xy.x\top y)(\lambda zu.z)A \longrightarrow (\lambda zu.z)\top A \longrightarrow \top$$

- Conditionals and pairs in λ-calculus:

$$\textbf{if } A \textbf{ then } P \textbf{ else } Q := APQ$$
$$\textbf{fst} := \lambda x.x\top, \qquad \textbf{snd} := \lambda x.x\bot, \qquad (P, Q) := \lambda x.xPQ$$

Example

$$\textbf{if } \top \textbf{ then } P \textbf{ else } Q \equiv \top PQ \to (\lambda xy.x)PQ \to P$$

Arithmetic

- Church's numerals (arithmetics on the Nat set):

$$
\begin{array}{lcl}
\underline{0} & := & \lambda fx.x \\
\underline{1} & := & \lambda fx.fx \\
\underline{n} & := & \lambda fx.f^n x \\
\textbf{add} & := & \lambda xypg.xp(ypq) \\
\textbf{mult} & := & \lambda xyz.x(yz) \\
\textbf{succ} & := & \lambda xyz.y(xyz) \\
\textbf{exp} & := & \lambda xy.yx \\
\textbf{iszero} & := & \lambda n.n(\top\bot)\top
\end{array}
$$

- **add** $\underline{n}\ \underline{m} \;=_\beta\; \underline{n+m}$
- **mult** $\underline{n}\ \underline{m} \;=_\beta\; \underline{n \times m}$

Exercise.

Expressiveness

In the mid 1930s

- **(Kleene)** Equivalence of λ-calculus and recursive functions
- **(Turing)** Equivalence of λ-calculus and Turing machines
- **(Curry)** Equivalence of λ-calculus and Combinatory Logic

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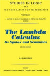
H.P. Barendregt. 

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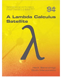
H.P. Barendregt, G. Manzonetto 

A Lambda Calculus Satellite

College Publications, 2022.

Roadmap

Lambda Calculus

$\lambda \rightarrow$ - Simple types

$\lambda \rightarrow$ simple (functional) types

Motivation

- "Disadvantages" of the untyped λ-calculus:
  - infinite computation - there exist λ-terms without a normal form
  - meaningless applications - it is allowed to create terms like **sin log**
- Types are syntactical objects that can be assigned to λ-terms
  - Reasoning with types was present in the early work of Church on untyped lambda calculus
- two typing paradigms:
  - *à la* Church - explicit type assignment (*typed lambda calculus*).
  - *à la* Curry - implicit type assignment (*lambda calculus with types*)

$\lambda \rightarrow$ syntax of types

$$\sigma \; ::= \; \alpha \mid (\sigma \rightarrow \sigma)$$

α ranges over TVar, a countable set of type variables

Conventions for minimising the number of the parentheses:

- $\sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_3$ stands for $(\sigma_1 \rightarrow (\sigma_2 \rightarrow \sigma_3))$

$$M : \sigma$$

Definition

- **Type assignment** is an expression of the form $M : \sigma$, where M is a λ-term and σ is a type
- **Declaration** $x : \sigma$ is a type assignment in which the term is a variable
- **Basis (context, environment)** $\Gamma = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ is a set of declarations in which all term variables are different
- **Statement (sequent)** $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \sigma$ ($\Gamma \vdash M : \sigma$)

$\lambda \rightarrow$ - the type system

à la Church and à la Curry

- Axiom

(*Ax*) $$\overline{\Gamma, x : \sigma \vdash x : \sigma}$$

- Rules

$(\rightarrow_{elim})$ $$\frac{\Gamma \vdash M : \sigma \rightarrow \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma . M : \sigma \rightarrow \tau} \; (\rightarrow_{intr}) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x . M : \sigma \rightarrow \tau}$$

à la Church   à la Curry

Running example: types

M	Type
xyz	$x : \sigma \to \tau \to \rho, y : \sigma, z : \tau \vdash xyz : \rho$
$\lambda x.zx$	$z : \sigma \to \rho \vdash \lambda x.zx : \sigma \to \rho$
$\mathbf{I} \equiv \lambda x.x$	$\sigma \to \sigma$
$\mathbf{K} \equiv \lambda xy.x$	$\sigma \to \rho \to \sigma$
$\mathbf{S} \equiv \lambda xyz.xz(yz)$	$\sigma \to \rho \to \tau \to (\sigma \to \tau) \to (\sigma \to \rho)$
$\Delta \equiv \lambda x.xx$	NO
$\mathbf{Y} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$	NO
$\Omega \equiv \Delta\Delta \equiv (\lambda x.xx)(\lambda x.xx)$	NO

I. $\lambda \rightarrow$ Fundamental properties

- Uniqueness of types
  If $\Gamma \vdash M : \sigma$ and $\Gamma \vdash M : \tau$, then $\sigma = \tau$
- Church-Rosser property holds in $\lambda \rightarrow$
- Subject reduction, type preservation under reduction

$$\text{If } M \longrightarrow P \text{ and } M : \sigma, \text{ then } P : \sigma.$$

  - Broader context: evaluation of terms (expressions, programs, processes) does not cause the type change.
  - type soundness
  - type safety = progress and preservation

II. $\lambda \rightarrow$ Strong normalisation

- Strong normalization

  If *M* : σ, then *M* is strongly normalizing.

  - Tait 1967
  - reducibility method (reducibility candidates, logical relations)
  - arithmetic proofs

III. $\lambda\to$ expressiveness

Selfapplication is not typable $\nvdash \lambda x.xx : \sigma$

Numerals are typeable

$\underline{n} \equiv \lambda f.\lambda x.f^n x : (\alpha\to\alpha)\to\alpha\to\alpha$ (exercise)

$\underline{n} \equiv \lambda x.\lambda f.f^n x : \alpha\to(\alpha\to\alpha)\to\alpha$ (exercise)

Definition (Extended polynomials)

The smallest class of functions over $\mathbb{N}$

- constant functions 0 and 1
- projections
- addition
- multiplication
- $\mathit{ifzero}(n, m, p) := \mathit{if}\ n = 0\ \mathit{then}\ m\ \mathit{else}\ p$

closed under composition

Theorem

M is typeable in $\lambda\to$ if and only if M is an extended polynomial

$\lambda \to$ and logic

Intuitionistic logic (minimal) - Natural deduction, Gentzen 1930s

- Axiom

(*Ax*) $$\frac{}{\Gamma, \sigma \vdash \sigma}$$

- Rules

$(\to_{elim})$ $$\frac{\Gamma \vdash \sigma \to \tau \qquad \Gamma \vdash \sigma}{\Gamma \vdash \tau}$$

$(\to_{intr})$ $$\frac{\Gamma, \sigma \vdash \tau}{\Gamma \vdash \sigma \to \tau}$$

$\lambda \to$ and logic

Intuitionistic logic (minimal) - Natural deduction, Gentzen 1930s

- Axiom

(*Ax*) $$\frac{}{\Gamma, x : \sigma \vdash x : \sigma}$$

- Rules

($\to_{elim}$) $$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

($\to_{intr}$) $$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

IV. $\lambda \rightarrow$ Curry-Howard correspondence

Intuitionistic logic vs computation

$$\vdash \sigma \quad \Leftrightarrow \vdash M : \sigma$$

A formula is provable in minimal intuitionistic logic
if and only if it is inhabited in $\lambda \rightarrow$.

- 1950s Curry
- 1968 (1980) Howard formulae-as-types

- 1970s Lambek - CCC Cartesian Closed Categories
- 1970s de Bruijn AUTOMATH

- 1970s Martin-Löf Type Theory

formulae (propositions)	–as–	types
proofs	– as –	terms
proofs	–as–	programs
proof normalisation	–as–	term reduction

- BHK - Brouwer, Heyting, Kolmogorov interpretation of logical connectives is formalized by the Curry-Howard correspondence

3 Type?

Type checking: given M and σ

$$(M : \sigma)?$$

Type inference (typability, type synthesis): given M

$$M : ?$$

Type inhabitation (term, program synthesis) : given σ

$$? : \sigma$$

$\lambda \rightarrow$ 3 Type?

Theorem

In $\lambda \rightarrow$

- *Type checking ($(M : \sigma)$?) is decidable*
- *Type inference (M :?) is decidable*
- *Type inhabitation (? : σ) is decidable*

$\lambda\to$ sum up

Advantages

- All terms are SN
- Typability, inhabitation, type checking decidable
- Types exactly all extended polynomials

Shortcomings

- no self-application
- no recursion
- no factorial
- no total functions
- not Turing complete

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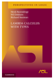
H.P. Barendregt, W. Dekkers, R. Statman 
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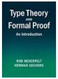
R. Nederpelt, H. Geuvers 

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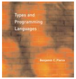
B. C. Pierece 

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