Lambda-Calculi for Logics — Valeria De Paiva

Lecture 2 - *June 24, 2025*

1 Classical Modal Logic

- Modal logic is the most successful logical framework in CS
- $\Box A = A$ is necessarily the case, A holds all of the time/at every world, etc.
- $\Diamond A = A$ is possibly the case, A holds at some time/some world, etc.
- Temporal logic, knowledge operators, BDI models, denotational semantics, effects, security modeling and verification, databases, etc. built off of classical modal logic - until about 20 years ago, it was the main modality considered

2 Constructive Reasoning

- If I ask you 'is there an x such that $P(x)$'?, I'm happier with the constructive reasoning answer 'yes, x_0' than with the classical reasoning answer 'yes, for all x it is not the case that not $P(x)$' (in other words, specialization rather than $\neg\forall x.(\neg P(x))$)
- Why: want reasoning to be as precise and safe as possible. To do this, use constructive reasoning as much as possible, classical if need be, but tell me where (classical can give a good indication for which direction to go for)
- Today is more about constructive modality

3 Intuitionistic Modal Logic

Basic idea: Modalities over intuitionistic propositional basis:

$$\wedge, \vee, \implies, \neg$$

Start here, then build. We want to put constructive modalities on top of constructive bases. Some questions arise:

- How do we choose the best constructive modalities? Certain properties of certain logics work better with certain systems.
- Which intuitionistic basis should we choose?
- How do modalities/systems relate to each other?
- Why are there so many modal logics?
- Which theorems should we work hard to preserve?
- Which are the most useful applications?

Simpson's 1994 PhD thesis is a good place for an overview of intuitionistic modal logic[1].

4 Constructive Modal Logic

Constructive logic is a logical basis for programming via Curry-Howard correspondences. A lot of places use 'constructive logic' and 'intuitionistic logic' interchangeably, but constructive logic is a broader term that includes intuitionistic logic and other logics. While intuitionistic logic determines the truth of a proposition by the existence of a proof, constructive logic focuses on the existence of a witness for a proposition[2]. Modalities are useful in computing, so constructive modalities ought to be twice as useful. But which constructive modalities should we use? Have we "gotten to the right one" yet?

- Usual pheonmenon: classical facts can be 'constructivized' in many different ways depending on expected behavior and desired tools. Hence constructive notions multiply which leads to too many choices. Our goal is to extract the best system we can.
- Operators $\Box$ and $\Diamond$ (like $\forall/\exists$) are not interdefinable, but they shouldn't be *completely* separate since they are related.
- The proof theory of modal logic is difficult, so we sometimes don't end up with the logics we expect/want.
- In finding good systems of constructive modal logic, lots of extra syntax/semantics have been added: hypersequents, labelled deduction systems, (linear) nested sequents, tree-sequents...

5 Intuitionistic Modal Logic and Applications (IMLA)

IMLA is a loose association bringing together researchers in the fields of philosophy, math, logic, and computer science in the hopes that they would be able to share their perspectives and benefit from others' research on intuitionistic modal logics and modal type theories. This has been semi-successful, but in the end, the researchers' different backgrounds leads people to often talk past one another.

IMLA had some early successes using CS4 and Lax logics, but due to the reasons detailed in the rest of these notes, some of the gains in modal type theories ended up not working out as well as expected.

6 Constructive S4 (CS4)

- Better behaved modal system, used by Gödel and Girard
- Goal: think linearly in certain cases and classically in others.
- Usual intuitionistic axioms plus MP, Nec rules and:

Modal Axioms[3][4]

- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\Box A \to A$
- $\Box A \to \Box\Box A$
- $\Box(A \to \Diamond B) \to (\Diamond A \to \Diamond B)$
- $A \to \Diamond A$ (If something is true than it is possibly true)

7 CS4 Sequent Calculus

S4 modal sequent rules first discussed in 1957 by Ohnishi and Matsumoto:

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}$$

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

$$\frac{\Box\Gamma, A \vdash B}{\Box\Gamma, \Diamond A \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}$$

Cut-elimination works, for classical and intuitionistic basis.

8 CS4 Natural Deduction

Natural Deduction (ND) has a more complicated rule

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

Abramsky's computational interpretation of linear logic leads to calculus that does not satisfy substitution[5].

Given proofs:

$$\dfrac{\dfrac{\Box A_1 \qquad \Box A_2}{B}}{\Box B} \qquad\qquad \dfrac{C \to \Box A_1 \qquad C}{\Box A_1}$$

We cannot derive $\Box B$ from $C \to \Box A_1, \Box A_2$ and C by combining the two proofs because $\Box B$ requires $\Box C$ in the context, which is not the case for the second proof.

One solution builds the substitutions into the rule as:

$$\frac{\Gamma \vdash \Box A_1, ..., \Gamma \vdash \Box A_k \Box A_1, ..., \Box A_k \vdash B}{\Gamma \vdash \Box B}(\Box I)$$

Another solution: Prawitz uses a notion of "essentially modal subformula" to guarantee substitutitivity in his monograph and makes several improvements. Overall, CS4 is "very well behaved" but still has issues.

9 CS4: Properties/Theorems

- Axioms satisfy deduction theorem, are equivalent to sequents,
- Sequents satisfy cut-elimination, sub-formula propery
- ND is equivalent to sequents
- ND satisfies normalization, ND assigns λ-terms curry howard equivalent
- Categorical model: monoidal comonad plus box-strong monad

Issues:

- Idempotency of comonad ($\Box A = \Box\Box A$) is not warranted and causes problems. Ideally, proofs should not be isomorphic - proving both directions is OK, but keeping them separate is much better.

- Rules are impure: in a couple of cases, to introduce $\Box$ on the right, you need to already know it on the left.

10 Dual Intuitionistic and Modal Logic

- Following LL can define a dual system for $\Box$-only modal logic
- Less impurity on rules, less commuting conversions, but what about $\Diamond$? what about other modal systems?

11 Lax Logic

Using Curry-Howard "backwards" to get the logic by dropping the terms

Motivation: Moggi's computational lambda calculus, an intuitionistic modal metalanguage for denotational semantics for programming language features: non-termination, differing evaluation strategies, non-determinism, side-effects are examples

11.1 Lax Modality Axioms

$A \to \Diamond A$ $\Diamond\Diamond A \to \Diamond A$ $(A \to B) \to (\Diamond A \to \Diamond B)$

12 System Lax

Also called CL-logic (for computational lambda calculus)

12.1 Lax Logic Properties

- Axioms, sequents, and ND are equivalent
- Deduction theorem holds, as does substitution and subject reduction
- The term calculus associated is strongly normalizing
- The reduction system given is confluent
- Cut elimination holds
- Lax logic (PLL) categorical mods as expected

13 Constructive K

Although Lax Logic and CS4 both have all important properties hold, modalities built on top of them are more difficult. We want properties like $\Diamond(A \vee B) \cong \Diamond A \vee \Diamond B$ and $\Diamond\bot \cong \bot$ (derivable from one another) which CS4 doesn't allow.

- Constructive K comes from proof-theoretical intuitions provided by Natural Deduction formulations of logic
- Note: only one rule for each connective, also $\Diamond$ depends on $\Box$

13.1 Properties

- Dual-context only for Box fragment
- For box-fragment OK. Have subject reduction, normalization and confluence for associated lambda-calculus
- Have categorical models, but too constrainted?
- Kripe semantics OK

Constructive K ends up having a "disturbing" non-uniformity of systems, so more work is necessary.

14 Alternatives

Many alternatives exist, but the main one that is used is described by Simpson [1]

- Simpson 1994 PhD: robust system for geometric ND theories for intuitionistic modal logic
- Justified by translation into intuitionistic first-order, recovers many of the systems in the literature
- Strong normalization and confluence proved for all the systems
- Normalization establishes completeness of cut-free sequent calculi and decidability of some of the systems
- decidable systems satisfy "finite model property"

The main problem with Simpson and the other alternatives is the lack of categorical models.

15 Wanted

- Constructive modal logics with axioms, sequents and natural deduction formulations, proved equivalent
- Cut-elimination, finite model property, (strong) normalization, confluence, and decidability
- Algebraic, Kripke, and categorical semantics
- Translating proofs more than simply theorems
- A broad view of constructive and/or modality
- If possibile, limitative results

16 Avron's Desiderata

- Should be able to handle a great diversity of logics, especially the traditional ones
- Should be independent of any particular semantics
- Structures should not be too complicated, yield a "real" subformula property
- Rules of inference should have a small fixed number of premises, and a local nature of application
- Should give us a better understanding of the logics and the differences between them

17 Diamond over Disjunction

Distribution of possibility over disjunction binary and nullary: Holds in IS4 (Simpson), but not in CS4

$$\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$$

$$\Diamond\bot \rightarrow \bot$$

- Distribution is canonical for classical modal logics, but many constructive modal logics don't satisfy it.
- Consequence: adding excluded middle gives you back classical modal logic

18 Labelled vs Unlabelled

- Introduction rule for $\Box$ says if A holds at every world y visible from x then $\Box A$ holds at x.
- Simpson two kinds of hypotheses: x : A means that the modal formula A is true in the world x; xRy, which says that world y is accessible from world x
- How reasonable is it to have your proposed semantics as part of your syntax?
- Proof-theoretic properties achieved, but no categorical semantics

References

[1] Alex K. Simpson. "The proof theory and semantics of intuitionistic modal logic". In: 1994. URL: https://api.semanticscholar.org/CorpusID:2309858.

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[3] Philippe Balbiani et al. *Constructive S4 modal logics with the finite birelational frame property*. 2024. arXiv: 2403.00201 [cs.LO]. URL: https://arxiv.org/abs/2403.00201.

[4] Natasha Alechina et al. "Categorical and Kripke Semantics for Constructive S4 Modal Logic". In: *Computer Science Logic*. Ed. by Laurent Fribourg. Berlin, Heidelberg: Springer Berlin Heidelberg, 2001, pp. 292–307. ISBN: 978-3-540-44802-0.

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