LAMBDA CALCULI THROUGH THE LENS OF LINEAR LOGIC

Lesson 1: **Linear Logic Proof-Nets**

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Agenda for Today

Agenda

MLL Formulas

Atomic Formulas: p and $\underline{p}$.

Formulas: $A ::= p \mid \underline{p} \mid A ⅋ B \mid A \otimes B$

(Involutory) Negation:

$$
\begin{array}{lll lll}
p^\perp & := & \underline{p} & \underline{p}^\perp & := & p \\
(A ⅋ B)^\perp & := & A^\perp \otimes B^\perp & (A \otimes B)^\perp & := & A^\perp ⅋ B^\perp
\end{array}
$$

Remark $(A^\perp)^\perp = A$

MLL (Two-Sided) Sequent Presentation

$$\frac{}{A \vdash A}\ (\mathtt{ax}) \qquad \frac{\Gamma \vdash A, \Delta \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ (\mathtt{cut})$$

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\ (\mathtt{perm}\ \mathsf{L}) \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ (\mathtt{perm}\ \mathsf{R})$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\ (\mathtt{par}\ \mathsf{L}) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ (\mathtt{par}\ \mathsf{R})$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ (\mathtt{tensor}\ \mathsf{L}) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ (\mathtt{tensor}\ \mathsf{R})$$

MLL (Unilateral) Sequent Presentation

$$\frac{}{\vdash A^\perp, A}\,(\texttt{ax}) \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,(\texttt{cut}) \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,(\texttt{perm})$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,(\texttt{par}) \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,(\texttt{tensor})$$

Agenda

How many proofs ending in $\vdash (A_1 \parr A_2), \ldots, (A_{2n-1} \parr A_{2n})$
if we start from a derivation of $\vdash A_1, \ldots, A_{2n}$?

If $n = 2$, then

$$\dfrac{\dfrac{\vdash A_1, A_2, A_3, A_4}{\vdash A_1 \parr A_2, A_3, A_4}\,(\mathtt{par})}{\vdash A_1 \parr A_2, A_3 \parr A_4}\,(\mathtt{par}) \qquad \dfrac{\dfrac{\vdash A_1, A_2, A_3, A_4}{\vdash A_1, A_2, A_3 \parr A_4}\,(\mathtt{par})}{\vdash A_1 \parr A_2, A_3 \parr A_4}\,(\mathtt{par})$$

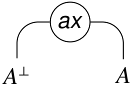
- $n!$: any possible sequent derivation captures a particular constructor **history** (there are different possible **sequentializations**).

Is there any **better representation** of proof derivations making abstraction of such **bureaucracy**?

MLL Proof-Nets

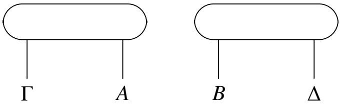
An **MLL Proof-Net (PN)** with conclusions $A_1, \ldots, A_n$ is a graph defined by induction as follows:

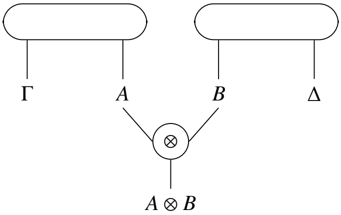
- For every MLL formula A, we have a **PN** with conclusions $A^\perp, A$ having the form:



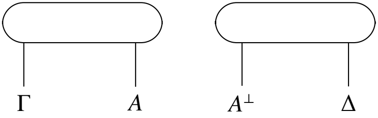
- Given a **PN** with conclusions Γ, A, B on the left, we can construct the **PN** with conclusions $\Gamma, A \parr B$ on the right.

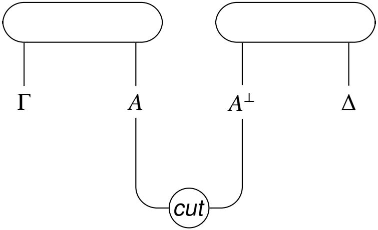
- Given a **PN** with conclusions Γ, A and a **PN** with conclusions Δ, B on the left, we can construct the **PN** with conclusions $\Gamma, A \otimes B, \Delta$ on the right.





- Given a **PN** with conclusions Γ, A and a **PN** with conclusions $\Delta, A^{\perp}$ on the left, we can construct the **PN** with conclusions Γ, Δ on the right.





Which is **PN** associated to the following proofs?

$$\dfrac{\dfrac{\vdash A_1, A_2, A_3, A_4}{\vdash A_1 \parr A_2, A_3, A_4}\,(\mathtt{par})}{\vdash A_1 \parr A_2, A_3 \parr A_4}\,(\mathtt{par}) \qquad \dfrac{\dfrac{\vdash A_1, A_2, A_3, A_4}{\vdash A_1, A_2, A_3 \parr A_4}\,(\mathtt{par})}{\vdash A_1 \parr A_2, A_3 \parr A_4}\,(\mathtt{par})$$

Digression

- No bureaucracy (the order of application of independent rules is abstracted away).

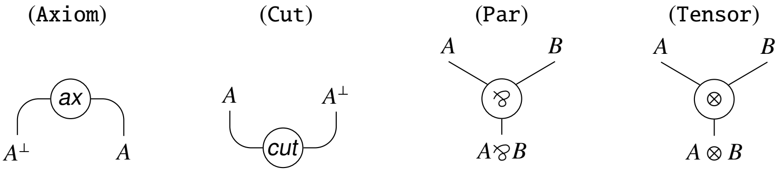
- No trace of the sequentialization/constructor history.

Closing the gap: how do we check that a given graph-like structure truly corresponds to a valid proof-net?

Agenda

Graph-Like Structures: Pre Proof-Nets

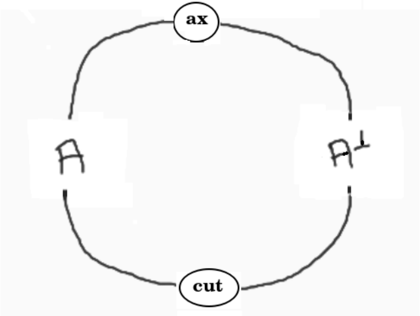
Pre Proof-Nets (Pre-PN) are generated by the following links:

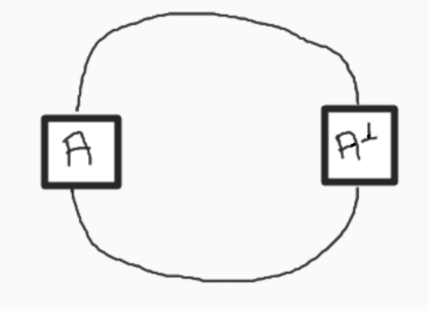


and satisfy the following conditions:

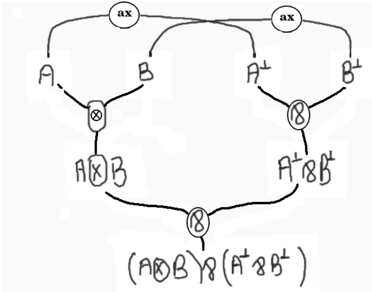
- Every formula is the conclusion of exactly one link.
- Every formula is the premise of at most one link.

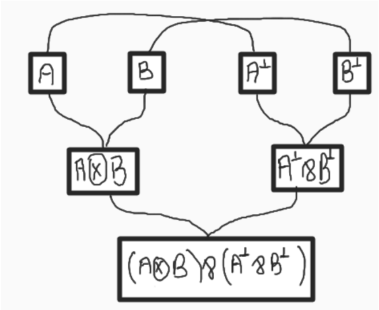
A Pre Proof-Net which is not a Proof-Net





A Pre Proof-Net which is a Proof-Net





Agenda

We need some correctness criteria on graph-like structures to check if they truly correspond to valid proof-nets

Some References

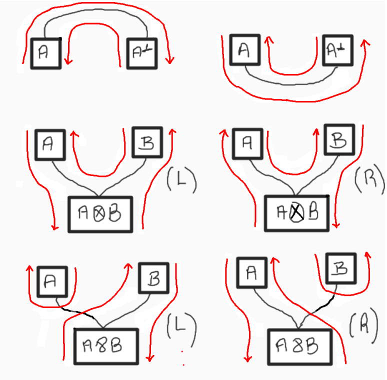
- Long-trip (Girard 87)
- Contractibility (Danos 90)
- Acyclic-Connected (Danos 90)
- Graph Parsing (Guerrini, Martini, Masini 97)
- ...

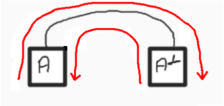
Long-Trip Criteria - Definitions

- Each link is a router:
  - The **ports** of the link are the associated formulas (premisses or conclusions)
  - The **routing rules** of the link indicate the exit port depending on the enter port

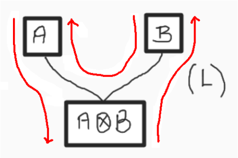
Moreover,
  - The axiom and cut links have only one possible **configuration**
  - The tensor and par links have two possible **configurations** (L) and (R)
  - The leaves connect the enter port with the exit port

Configurations

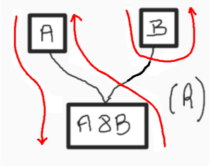




- A path entering at A, exits at $A^\perp$.
- A path entering at $A^\perp$, exits at A.



- A path entering at B, exits at A.
- A path entering at $A \otimes B$, exits at B.
- A path entering at A, exits at $A \otimes B$.



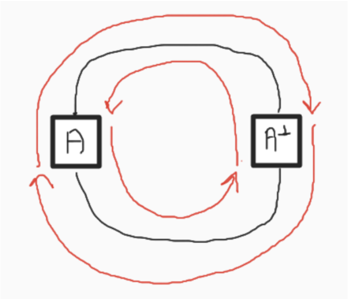
- A path entering at B, exits at B.
- A path entering at A, exits at $A \parr B$.
- A path entering at $A \parr B$, exits at A.

Theorem

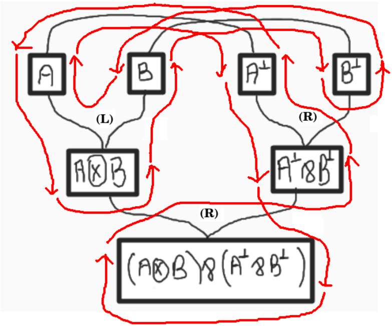
A Pre-PN is a **PN** *iff for* **every possible configuration** *of the links there is a unique* **long trip** *(i.e. a cycle visiting each node exactly once in each direction).*

Exponential Complexity

A Pre-PN which is not a **PN**



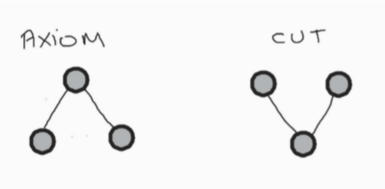
A Pre PN which is a **PN**

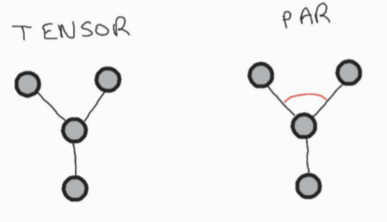


Acyclic Connected Criteria (ACC) - Definitions

- A Pre-PN is abstracted by a **paired graph** S equipped with
  - A set V of vertices
  - A set E of edges
  - A set $C(S)$ of pairwise disjoint **pairs** of co-incident (*i.e.*one node in common) **edges**. They are marked with an arc in **red**.

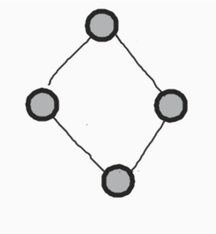
For every Pre-PN P we define a paired graph P^{-} by using the following constructions:



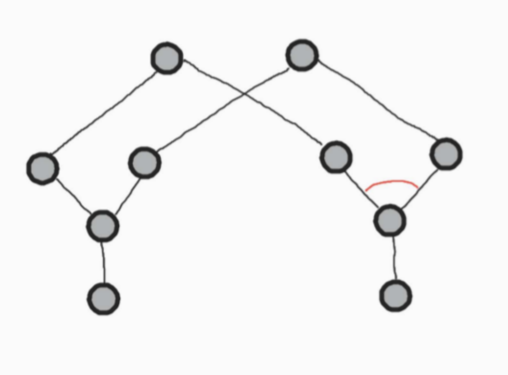


Examples

Paired Graph **P1**



Paired Graph **P2**

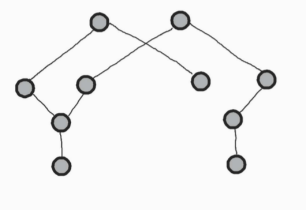


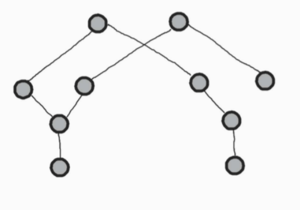
Removing Edges

Definition

Let S be a paired graph. Then $R(S)$ is the set of graphs obtained from S by removing **exactly** one edge for **each** pair in $C(S)$.

Coming back to the paired graph **P2**:





Acyclic Connected Criteria (ACC)

Remark: Every Pre-PN P can be seen (*i.e.* abstracted) as a paired graph P^-.

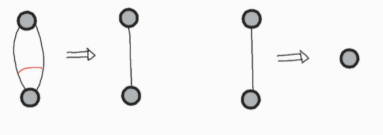
Theorem

A Pre-PN P is a PN iff every graph in $R(P^-)$ is connected and acyclic (i.e. a tree).

Still Exponential: 2^p, where p is the number of par links in the graph

Contractibility - Definitions

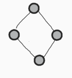
We consider the following rewriting system with two rules:



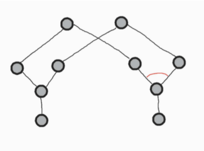
Given a paired graph S:

- The first rule can only be applied to a pairwise disjoint pair of edges of $C(S)$ connecting the **same** two nodes.
- The second rule can only be applied to an edge (not in $C(S)$) connecting two **different** nodes.

Examples



reduces to 



reduces to 

Theorem

A Pre-PN P is a PN iff P^{-} is **contractible** *(i.e. reduces to a single node).*

Quadratic Time

Agenda

MELL Formulas

Atomic Formulas: p and $\underline{p}$.

Formulas: $A ::= p \mid \underline{p} \mid A \parr B \mid A \otimes B) \mid ?A \mid !A$

(Involutory) Negation:

$$
\begin{array}{lcllcl}
p^{\perp} & := & \underline{p} & \underline{p}^{\perp} & := & p \\
(A \parr B)^{\perp} & := & A^{\perp} \otimes B^{\perp} & (A \otimes B)^{\perp} & := & A^{\perp} \parr B^{\perp} \\
(?A)^{\perp} & := & !A^{\perp} & (!A)^{\perp} & := & ?A^{\perp}
\end{array}
$$

Remark $(A^{\perp})^{\perp} = A$

MELL (Two-Sided) Sequent Presentation

$$\frac{}{A \vdash A}\ (\texttt{ax}) \qquad \frac{\Gamma \vdash A, \Delta \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ (\texttt{cut}) \qquad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\ (\texttt{perm}\ \mathsf{L}) \quad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ (\texttt{perm}\ \mathsf{R})$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\ (\texttt{par}\ \mathsf{L}) \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ (\texttt{par}\ \mathsf{R})$$

$$\frac{\Gamma, A, B, \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ (\texttt{tensor}\ \mathsf{L}) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ (\texttt{tensor}\ \mathsf{R})$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\ \texttt{weak}) \quad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\ \texttt{cont}) \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\ \texttt{der}) \quad \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}\ (?\ \mathsf{L})$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\ (?\ \texttt{weak}) \quad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\ (?\ \texttt{cont}) \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta}\ (?\ \texttt{der}) \quad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ (!\ \mathsf{R})$$

MELL (Unilateral) Sequent Presentation

$$\frac{}{\vdash A^{\perp}, A}\,(\mathtt{ax}) \qquad \frac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(\mathtt{cut}) \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,(\mathtt{perm})$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,(\mathtt{par}) \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,(\mathtt{tensor})$$

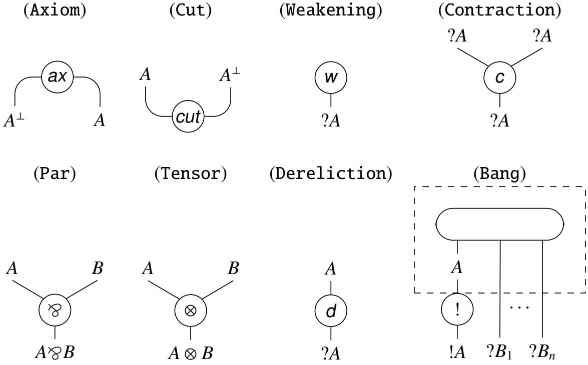
$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,(\mathtt{weak}) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,(\mathtt{cont}) \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,(\mathtt{der}) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,(\mathtt{bang})$$

Example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A, A^{\perp}$}
\RightLabel{(der)}
\UnaryInfC{$\vdash A, ?A^{\perp}$}
\RightLabel{(par)}
\UnaryInfC{$\vdash A \parr ?A^{\perp}$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A, A^{\perp}$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A, A^{\perp}$}
\RightLabel{(der)}
\UnaryInfC{$\vdash A, ?A^{\perp}$}
\RightLabel{(bang)}
\UnaryInfC{$\vdash !A, ?A^{\perp}$}
\RightLabel{(tensor)}
\BinaryInfC{$\vdash A^{\perp} \otimes !A, A, ?A^{\perp}$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash A, ?A^{\perp}$}
\RightLabel{(weak)}
\UnaryInfC{$\vdash A, ?A^{\perp}, ?A^{\perp}$}
\RightLabel{(cont)}
\UnaryInfC{$\vdash A, ?A^{\perp}$}
\RightLabel{(weak)}
\UnaryInfC{$\vdash A, ?A^{\perp}, ?C^{\perp}$}
\end{prooftree}
$$

Agenda

MELL Pre-PN



(Axiom)
(Cut)
(Weakening)
(Contraction)
ax
$A^\perp$
A
cut
A
$A^\perp$
w
$?A$
$?A$
$?A$
c
$?A$
(Par)
(Tensor)
(Dereliction)
(Bang)
A
B
$A \parr B$
A
B
$A \otimes B$
A
d
$?A$
A
!
$\cdots$
$!A$
$?B_1$
$?B_n$

Intuition Behind MELL Proof-Nets

Erasable/**Duplicable** formulas are captured inside **BOXES**.

MELL Proof-Nets

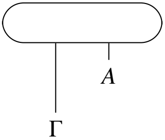
A **MELL Proof-Net (PN)** with conclusions $A_1, \ldots, A_n$ is defined by induction as follows:

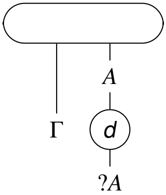
- Every MLL **PN** is a MELL **PN**.
- Given a **PN** with conclusions Γ on the left, we can construct the **PN** with conclusions $\Gamma, ?A$ on the right.

- Given a **PN** with conclusions $\Gamma, ?A, ?A$ on the left, we can construct the **PN** with conclusions $\Gamma, ?A$ on the right.

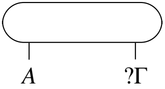
MELL Proof-Nets

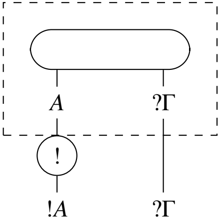
- Given a **PN** with conclusions Γ, A on the left, we can construct the **PN** with conclusions $\Gamma, ?A$ on the right.



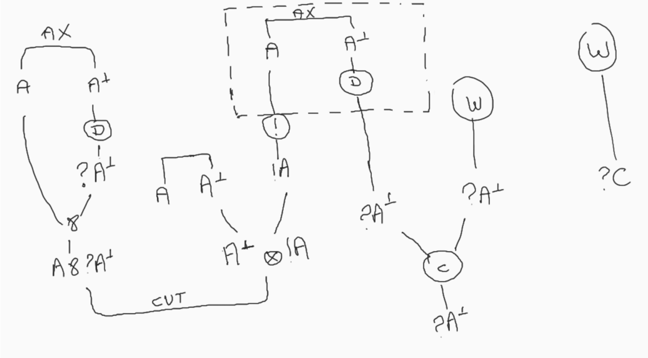


- Given a **PN** with conclusions $?\Gamma, A$, we can construct the following **PN** with conclusions $?\Gamma, !A$





Example

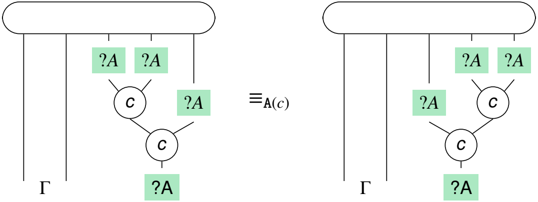


About the Operational Semantics

- **Structural transformations**:
  - Equivalences
  - Additional Rewriting Rules
- **Cut Elimination transformations**:
  - **Multiplicative** Rewriting Rules:
    Reduction rules not involving **BOXES**
  - **Exponential** Rewriting Rules:
    Reduction rules involving **BOXES**

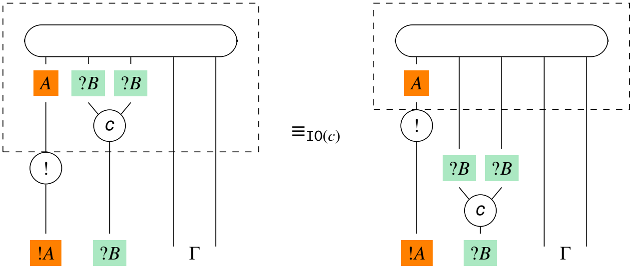
Equivalence Relation for MELL Proof-Nets

$$\dfrac{\dfrac{\vdash \Gamma, ?A^1, ?A^2, ?A^3}{\vdash \Gamma, ?A^{1,2}, ?A^3}\,(\mathtt{cont})}{\vdash \Gamma, ?A^{1,2,3}}\,(\mathtt{cont}) \qquad \dfrac{\dfrac{\vdash \Gamma, ?A^1, ?A^2, ?A^3}{\vdash \Gamma, ?A^1, ?A^{2,3}}\,(\mathtt{cont})}{\vdash \Gamma, ?A^{1,2,3}}\,(\mathtt{cont})$$



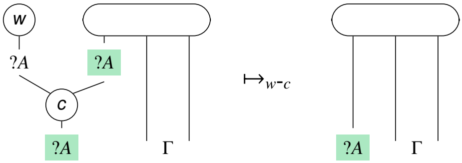
Equivalence Relation for MELL Proof-Nets

$$\dfrac{\dfrac{\vdash ?\Gamma, ?B, ?B, A}{\vdash ?\Gamma, ?B, A}\ (\mathtt{cont})}{\vdash ?\Gamma, ?B, !A}\ (\mathtt{bang}) \qquad \dfrac{\dfrac{\vdash ?\Gamma, ?B, ?B, A}{\vdash ?\Gamma, ?B, ?B, !A}\ (\mathtt{bang})}{\vdash ?\Gamma, ?B, !A}\ (\mathtt{cont})$$



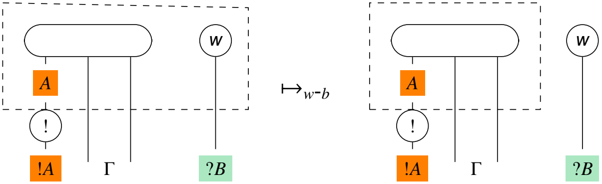
Additional Rewriting Rules for MELL Proof-Nets

$$\dfrac{\dfrac{\vdash \Gamma, ?A}{\vdash \Gamma, ?A, ?A}\;(\texttt{weak})}{\vdash \Gamma, ?A}\;(\texttt{cont}) \qquad\qquad \vdash \Gamma, ?A$$



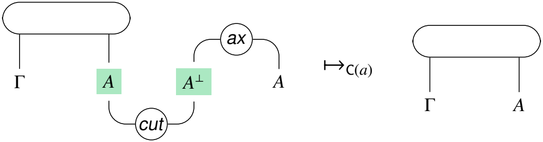
Additional Rewriting Rules for MELL Proof-Nets

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, A, ?B}\,(\mathtt{weak})}{\vdash ?\Gamma, !A, ?B}\,(\mathtt{bang}) \qquad \dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,(\mathtt{bang})}{\vdash ?\Gamma, !A, ?B}\,(\mathtt{weak})$$



Multiplicative Cut Elimination Rules for MELL Proof-Nets

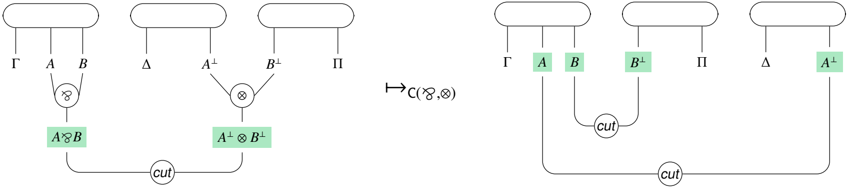
$$\dfrac{\vdash \Gamma, A \qquad \dfrac{}{\vdash A^{\perp}, A}\,(\mathtt{ax})}{\vdash \Gamma, A}\,(\mathtt{cut}) \qquad\qquad \vdash \Gamma, A$$



Multiplicative Cut Elimination Rules for MELL Proof-Nets

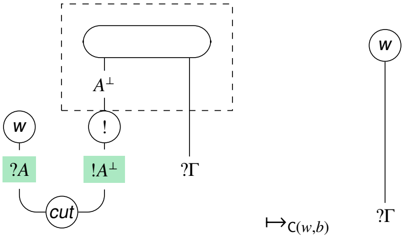
$$\dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(\mathtt{par}) \qquad \dfrac{\vdash \Delta, A^{\perp} \qquad \vdash \Pi, B^{\perp}}{\vdash \Delta, A^{\perp} \otimes B^{\perp}, \Pi}\,(\mathtt{tensor})}{\vdash \Gamma, \Delta, \Pi}\,(\mathtt{cut})$$

$$\dfrac{\dfrac{\vdash \Gamma, A, B \qquad \vdash \Pi, B^{\perp}}{\vdash \Gamma, \Pi, A}\,(\mathtt{cut}) \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta, \Pi}\,(\mathtt{cut})$$



Exponential Cut Elimination Rules for MELL Proof-Nets

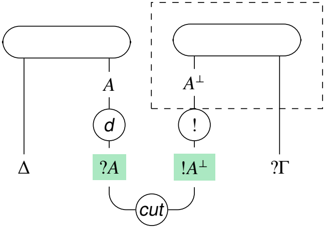
$$\dfrac{\dfrac{\vdash \Delta}{\vdash \Delta, ?A}\,(\mathtt{weak}) \qquad \dfrac{\vdash ?\Gamma, A^{\perp}}{\vdash ?\Gamma, !A^{\perp}}\,(\mathtt{bang})}{\vdash \Delta, ?\Gamma}\,(\mathtt{cut}) \qquad\qquad \dfrac{\vdash \Delta}{\vdash \Delta, ?\Gamma}\,(\mathtt{weak})$$



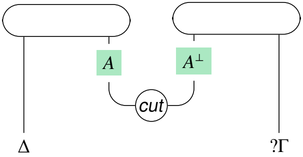
Exponential Cut Elimination Rules for MELL Proof-Nets

$$\dfrac{\dfrac{\vdash \Delta, A}{\vdash \Delta, ?A}\,(\texttt{der}) \qquad \dfrac{\vdash ?\Gamma, A^{\perp}}{\vdash ?\Gamma, !A^{\perp}}\,(\texttt{bang})}{\vdash \Delta, ?\Gamma}\,(\texttt{cut})$$

$$\dfrac{\vdash \Delta, A \qquad \vdash ?\Gamma, A^{\perp}}{\vdash \Delta, ?\Gamma}\,(\texttt{cut})$$



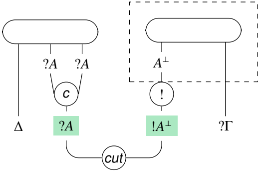
$\mapsto_{c(d,b)}$



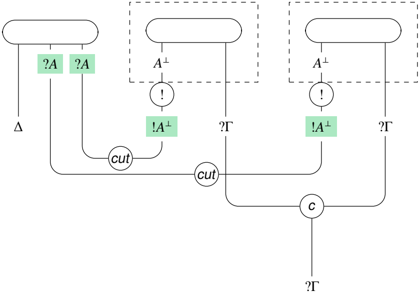
Exponential Cut Elimination Rules for MELL Proof-Nets

$$\dfrac{\dfrac{\vdash \Delta, ?A, ?A}{\vdash \Delta, ?A}\,(\mathtt{cont}) \qquad \dfrac{\vdash ?\Gamma, A^{\perp}}{\vdash ?\Gamma, !A^{\perp}}\,(\mathtt{bang})}{\vdash \Delta, ?\Gamma}$$

$$\dfrac{\dfrac{\dfrac{\vdash \Delta, ?A, ?A \qquad \dfrac{\vdash ?\Gamma, A^{\perp}}{\vdash ?\Gamma, !A^{\perp}}\,(\mathtt{bang})}{\vdash \Delta, ?\Gamma, ?A}\,(\mathtt{cut}) \qquad \dfrac{\vdash ?\Gamma, A^{\perp}}{\vdash ?\Gamma, !A^{\perp}}\,(\mathtt{bang})}{\vdash \Delta, ?\Gamma, ?\Gamma}\,(\mathtt{cut})}{\vdash \Delta, ?\Gamma}\,(\mathtt{cont})$$



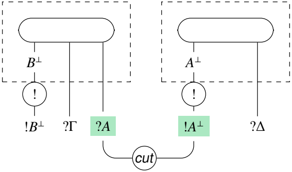
$\mapsto_{\mathsf{C}(c,b)}$



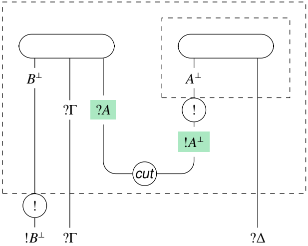
Exponential Cut Elimination Rules for MELL Proof-Nets

$$\dfrac{\dfrac{\vdash ?\Gamma, B^{\perp}, ?A}{\vdash ?\Gamma, !B^{\perp}, ?A}\,(\mathtt{bang}) \qquad \dfrac{\vdash ?\Delta, A^{\perp}}{\vdash ?\Delta, !A^{\perp}}\,(\mathtt{bang})}{\vdash ?\Gamma, ?\Delta, !B^{\perp}}\,(\mathtt{cut})$$

$$\dfrac{\dfrac{\vdash ?\Gamma, B^{\perp}, ?A \qquad \dfrac{\vdash ?\Delta, A^{\perp}}{\vdash ?\Delta, !A^{\perp}}\,(\mathtt{bang})}{\vdash ?\Gamma, ?\Delta, B^{\perp}}\,(\mathtt{cut})}{\vdash ?\Gamma, ?\Delta, !B^{\perp}}\,(\mathtt{bang})$$



$\mapsto_{\mathsf{C}(b,b)}$



The Reduction Relation for MELL Proof-Nets

Let consider the following relations:

$$\begin{array}{lcl} \mathcal{R} & := & \{\mathtt{C}(a), \mathtt{C}(\parr, \otimes), \mathtt{C}(w, b), \mathtt{C}(d, b), \mathtt{C}(c, b), \mathtt{C}(b, b), w\text{-}b, w\text{-}c\} \\ \mathcal{E} & := & \{\mathtt{A}(c), \mathtt{IO}(c)\} \end{array}$$

- The reduction relation $\rightarrow_{\mathcal{R}}$ is the closure by all **PN** contexts of the rules in $\mathcal{R}$.
- The congruence $\simeq_{\mathcal{E}}$ is the reflexive, symmetric, transitive, closed by **PN** contexts relation on **PN** generated by the equations $\mathcal{E}$.

Said differently, the reduction rules in $\mathcal{R}$ and congruences rules in $\mathcal{E}$ are applied locally **inside** some (common) context.

Finally, we shall write $\rightarrow_{\mathcal{R}/\mathcal{E}}$ for the reduction relation on MELL proof-nets generated by the reduction relation $\rightarrow_{\mathcal{R}}$ modulo the congruence $\simeq_{\mathcal{E}}$, *i.e.*

$$p \rightarrow_{\mathcal{R}/\mathcal{E}} p' \text{ iff } \exists p_1, p_2 \text{ such that } p \simeq_{\mathcal{E}} p_1 \rightarrow_{\mathcal{R}} p_2 \simeq_{\mathcal{E}} p'$$

Definition

A reduction relation $\mathcal{S}$ is said to be **confluent** if and only if for every t, u, v such that $t \rightarrow^*_{\mathcal{S}} u$ and $t \rightarrow^*_{\mathcal{S}} v$ there is t' such that $u \rightarrow^*_{\mathcal{S}} t'$ and $v \rightarrow^*_{\mathcal{S}} t'$.

Theorem (Confluence)

The reduction $\rightarrow_{\mathcal{R}/\mathcal{E}}$ is confluent on MELL Proof-Nets.

MELL Proof-Nets Properties

Definition

- A reduction relation $\mathcal{S}$ is said to be **terminating** if and only if for every t there is no infinite $\rightarrow_{\mathcal{S}}$ -sequence starting at t (*i.e.* every $\rightarrow_{\mathcal{S}}$ -reduction sequence starting at any term is terminating).
- A reduction relation $\mathcal{S}$ is said to be **strongly normalizing** if and only if every **typed object** t is terminating.

Theorem (Strong Normalization)

The reduction $\rightarrow_{\mathcal{R}/\mathcal{E}}$ is terminating on MELL Proof-Nets (i.e. $\rightarrow_{\mathcal{R}/\mathcal{E}}$ is strongly normalizing).

Proof.

Based on strong normalization of $\rightarrow_{\mathcal{R}}$ by Girard. □


THANK
YOU
FOR
YOUR
ATTENTION


Time for
Questions