Category theory

Q: What is category theory?

A: Study of objects based on how they interact with each other.
→ External perspective
→ Usually on concrete and nice spaces. Non-internal perspective.

A: Some definitions: some non-empty collection of objects (conf. sets)
Q: What is a category?

A: Ingredients:
- a collection $Ob(\mathcal{C})$ called objects and
- a collection of morphisms between objects

Ex. Category set: these objects: sets
morphisms: functions

Ex. The category Top: these objects: topological spaces
morphisms: continuous functions

Ex. The category Top: these objects: topos
morphisms: functions

Ex. Group: these objects: groups
morphisms: group homomorphisms

Vect: these objects: vector spaces
morphisms: linear transformations

Ex. Top/Top: these objects: topological spaces
morphisms: maps to topos, a topos mapping one topos into another. Preserving topos structure, structure.

Ex. Long Ass Group: these objects: topological groups
morphisms: continuous homomorphisms

...

More formally:

Def. A category $\mathcal{C}$ consists of
- a collection $Ob\,\mathcal{C}$ of the objects (usually denoted $X, Y, Z, \ldots$)

for each pair $X, Y \in Ob\,\mathcal{C}$ a collection $Hom(X,Y)$ of morphisms from X, Y
(usually denoted $f: X \to Y$ or $X \xrightarrow{f} Y$ or $X \xrightarrow{\quad} Y$)

together with
- for each object X, an identity morphism $id_X : X \to X$
- for each triple of objects X, Y, Z, composition $\circ : X \to Y$, $g: Y \to Z$, $h \circ g : X \to Z$
(from the above: $g \circ f : X \to Z$)

such that
- for every $f: X \to Y$, we have
$f \circ id_X = f = id_Y \circ f$ (left and right identity)
- for every $f: X \to Y$, $g: Y \to Z$, $h: Z \to W$, we have
$h \circ (g \circ f) = (h \circ g) \circ f$ (associativity)

Ex. (Identity) Identity Id_X. In Top given monoidality.

Ex. Associativity in Set: $g \circ f$ in Set, $g: U \to V$, $f: X \to U$
$(h \circ g) \circ f = h \circ (g \circ f)$
$(h \circ g) \circ f = \{(x, (h \circ g)(x)) \mid x \in X\}$
$= \{(x, h(g(x))) \mid x \in X\}$
$= \{(x, h(y)) \mid x \in X, y = f(x)\}$
$= \{(x, (h \circ g)(y)) \mid x \in X, y = f(x)\}$
$= h \circ (g \circ f)$ ✓

In Top: Say $Hom(T, S) = \{f \mid f: T \to S$ continuous$\}$
So $f \circ id_T = f = id_S \circ f$, $f: T \to S$
and $h \circ (g \circ f) = (h \circ g) \circ f$, $f: T \to S$, $g: S \to U$, $h: U \to V$
So $f \circ id_T = f = id_S \circ f$ and $h \circ (g \circ f) = (h \circ g) \circ f$ ✓

Ex. (Identity) Identity Id_X. In Top given monoidality.

Ex. Associativity in Set: $g \circ f$ in Top