

Category theory

Q: What is category theory?

A: Study of objects based on how they interact with each other.

→ External perspective

→ Usually as powerful and more than an internal perspective.

A: Since we take high-level view, can compare whole categories of things.
(Corry-Howard)

Q: What is a category?

A: Roughly:

- a collection (set, type, etc) of objects
- a collection of morphisms between objects

Ex. The category Set has objects: sets
morphisms: functions

Ex. The category Prop has objects: propositions
morphisms: implication / derivation

Ex. The category Ty has objects: types
morphisms: functions

Ex. Grp objects: groups
morphisms: group homomorphisms

Vect objects: vector spaces
morphisms: linear transformations

Ex. ProgSpec has objects: program specifications
morphisms: way to produce a program meeting one specification
from any program meeting another

Ex. Lang has objects: programming languages
morphisms: translations / compilations

...

More formally:

Def. A category \mathcal{C} consists of

- a collection $\text{ob } \mathcal{C}$ of the objects (usually denoted X, Y, Z, \dots)
 - for each pair $X, Y \in \text{ob } \mathcal{C}$ a collection $\text{hom}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y
(usually denoted $f: X \rightarrow Y$ or $g, h, \dots: X \rightarrow Y$)
- together with
- for each object X , an identity morphism $\text{id}_X: X \rightarrow X$
 - for each triple of objects X, Y, Z , morphisms $f: X \rightarrow Y$, $g: Y \rightarrow Z$, a composition morphism $g \circ f: X \rightarrow Z$
(note the order of $g \circ f$!)

such that

- for every $f: X \rightarrow Y$, we have
 $f \circ \text{id}_X = f = \text{id}_Y \circ f$ (left and right identity)
- for every $f: X \rightarrow Y$, $g: Y \rightarrow Z$, $h: Z \rightarrow A$, we have
 $h \circ (g \circ f) = (h \circ g) \circ f$ (associativity)

Ex. (identity) Usually $\lambda x.x$. In Prop, given axiomatically.

Ex. Associativity in Set: $g \circ f$ is $\lambda x.g(f(x))$
so $\text{id}_h \circ (g \circ f)(x) = \text{id}_h(g(f(x)))$
 $= h(g(f(x)))$
 $(h \circ g) \circ f = (h \circ g)(f(x))$
 $= h(g(f(x)))$ ✓

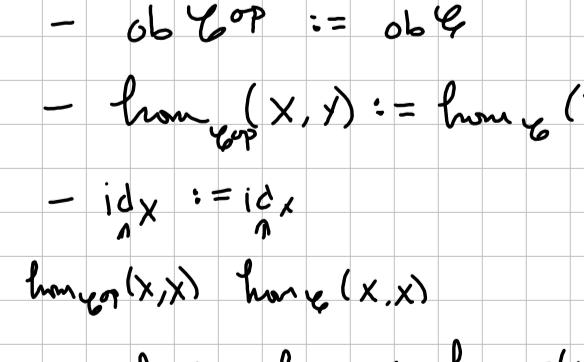
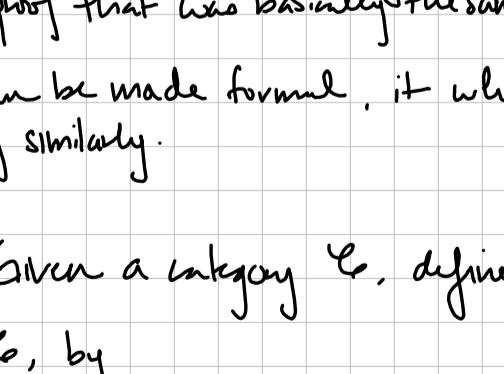
In Prop: Say $\text{hom}(P, Q) = \{f\}$ if $P = Q$
otherwise.

So if $T: P \rightarrow Q$, $T: Q \rightarrow R$, $T: R \rightarrow S$,
can derive first that $T: P \rightarrow R$ and then $T: P \rightarrow S$
or first that $T: Q \rightarrow S$ and then $T: P \rightarrow S$.

In either case it is $T: P \rightarrow S$.

- In other words $T \circ (T \circ T)$, $(T \circ T) \circ T: P \rightarrow S$
But there is (at most) one element in $P \rightarrow S$, so
these must be equal.

Notn. Category theory becomes more readable via diagrams.



This square/diagram commutes if $g \circ f = i \circ h$.

(Can write \circ , but usually omitted.)

See also string diagrams (MLL proof nets are very similar)

③

Terminal and initial objects

We not only understand objects by their relations, but define/specify them via their relations.

Def. Let \mathcal{C} be a category, and $X \in \mathcal{C}$. Say that X is terminal if there is exactly one morphism $Z \rightarrow X$ for all $Z \in \mathcal{C}$.

Ex. What is the terminal object of Set?

(These proofs are usually guess-and-check.)

The singleton is terminal - there is indeed exactly one map from any set to the singleton.

Ex. Prop?

T , the true proposition

Ex. Ty? unit

Isomorphism

We regard two objects in a category \mathcal{C} as "the same" if they are isomorphic. (We disregard equality.)

Def. Consider $f: X \rightarrow Y$ in a category \mathcal{C} . Say f is an isomorphism if there is another map $g: Y \rightarrow X$ s.t. $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Often write f^{-1} for g , $f: X \cong Y$, or $X \cong Y$. Say X and Y are isomorphic.

Ex. In Set isomorphisms are bijections.

In Prop isomorphisms are bimorphisms

Ex. In fact isomorphisms are bijections. In Set isomorphisms are bijections up to isomorphism and we thus talk about the terminal object if it exists.)

Def. Given a category \mathcal{C} , define the category \mathcal{C}^{op} , the opposite of \mathcal{C} , by

- $\text{ob } \mathcal{C}^{\text{op}} := \text{ob } \mathcal{C}$
- $\text{hom}_{\mathcal{C}^{\text{op}}}(X, Y) := \text{hom}_{\mathcal{C}}(Y, X)$

$$\text{id}_X := \text{id}_X$$

$$\text{hom}_{\mathcal{C}^{\text{op}}}(X, X) = \text{hom}_{\mathcal{C}}(X, X)$$

$$- g \circ f := f \circ g \in \text{hom}_{\mathcal{C}^{\text{op}}}(X, Z) = \text{hom}_{\mathcal{C}}(Z, X)$$

$$\text{in } \mathcal{C}^{\text{op}} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \xleftarrow{g} & \\ & Z & \end{array}$$

Ex. Check that this forms a category.

Ex. Check that if $X \cong Y$ in \mathcal{C} , then $X \cong Y$ in \mathcal{C}^{op} .

Initial objects

Def. Say that $I \in \mathcal{C}$ is initial if it is terminal in \mathcal{C}^{op} .

In other words, I is initial if for all $Z \in \mathcal{C}$, there is a unique map $I \rightarrow Z$.

Ex. Check these two definitions are equivalent.

Def. Initial objects are unique up to unique isomorphism.

④

If. This is the dual statement of the lemma above.

Ex. In Set, the initial object is \emptyset .

In Prop, the initial object is \perp .

In Ty, the initial object is empty.

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