OREGON PROGRAMMING LANGUAGES SUMMER SCHOOL

Introduction to Category Theory — Paige Randall North

Lecture 2 - *July 1, 2025*

1 Terminal Objects

Definition 1.1 (Terminal object)**.** An object T in a category $\mathcal{C}$ is a valid terminal if for every object $Z \in \mathsf{ob}\ \mathcal{C}$, there is exactly one map $Z \xrightarrow{!} T$.

Example 1.1. Consider the following example categories, as well as their terminal objects (or lack thereof).

1. $\mathcal{Set}$:

   The terminal object of the category $\mathcal{Set}$ is the singleton set $\{*\}$. For any set Z, we have exactly one function from Z to $\{*\}$:

   $$!_Z := \lambda z.* \quad : \quad Z \longrightarrow \{*\}$$

2. $\mathcal{Ty}$:

   The terminal object of the category $\mathcal{Ty}$ is the unit type `unit`. For any type Z, we have exactly one function from Z to `unit`:

   $$!_Z := \lambda z.\mathtt{tt} \quad : \quad Z \longrightarrow \mathtt{unit}$$

3. $\mathcal{Prop}$:

   The terminal object of the category $\mathcal{Prop}$ is the true proposition True. For any proposition P, we immediately have that it implies True. Additionally, we have only one such map by definition.

4. The category $\mathcal{C}$ with a single object A:

   The terminal object of the category is A, with the identity function as the only morphism to A.

   $$!_A := \mathrm{id}_A \quad : \quad A \longrightarrow A$$

5. The category $\mathcal{C}$ with two objects A, B and no maps (beyond the identity maps):

   This category does not have any terminal objects (there is no map from A to B or from B to A).

6. The category $\mathcal{C}$ with two objects A, B and the map $A \xrightarrow{f} B$:

   This terminal object of this category is B. We have the following unique maps:

$$\begin{array}{lcllc}
!_A & := & f & : & A \longrightarrow B \\
!_B & := & \mathrm{id}_A & : & B \longrightarrow B
\end{array}$$

2 Isomorphisms

In category theory, we typically do not consider equality between objects, but talk about equivalence up to isomorphism.

Definition 2.1 (Isomorphism)**.** A map $f : X \to Y$ is an isomorphism if there exists another map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If such a map exists, we say that X and Y are isomorphic.

Notation 2.1 (Isomorphism)**.** We write $X \cong Y$ to mean there exists an isomorphism between X and Y, $f : X \cong Y$ to mean f is an isomorphism, and f^{-1} to be the inverse of f.

Isomorphisms can also be represented with commuting diagrams:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

In $\mathcal{Set}$, isomorphisms are bijections between sets. In $\mathcal{Prop}$, isomorphisms are bi-implications between formulae.

Exercise 2.1. If f is an isomorphism, then f^{-1} is unique.
Hint: Use the definition, assume two inverse options g and h, show they must be equal.

Lemma 2.1. If $f : X \cong Y$, then for any $Z \in \mathsf{ob}\ \mathcal{C}$, we have the following.

1. $f_* : \mathsf{hom}(Z, X) \cong \mathsf{hom}(Z, Y)$
2. $f^* : \mathsf{hom}(X, Z) \cong \mathsf{hom}(Y, Z)$

This justifies thinking about sets (objects) up to bijection because we cannot distinguish between isomorphic objects.

Proof sketch. Define f_* by $(Z \xrightarrow{g} X) \mapsto (Z \xrightarrow{g} X \xrightarrow{f} Y)$, or $f_*(g) = f \circ g$. We show this with a diagram:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[rr, bend right, "f_*(g)"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

We also define f_*^{-1} by $(Z \xrightarrow{h} Y) \mapsto (Z \xrightarrow{h} Y \xrightarrow{f^{-1}} X)$. Diagrammatically:

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[rr, bend right, "f_*^{-1}(h)"'] & Y \arrow[r, "f^{-1}"] & X \end{tikzcd}$$

Do case analysis on composition of f_* and f_*^{-1} to see that the f_* terms cancel out. □

Lemma 2.2 (Uniqueness of terminal objects)**.** Terminal objects are unique up to unique isomorphism.

Proof. Consider two terminal objects $T, T' \in \mathsf{ob}\ \mathcal{C}$. Since T' is terminal, there exists a unique map $T \xrightarrow{!} T'$, similarly there exists a unique map $T' \xrightarrow{!'} Z$. Now the composition $! \circ !'$ given by $T \xrightarrow{!} T' \xrightarrow{!'} T$ must be equal to id_T by the requirements of T being terminal and the existence of id_T. The same reasoning applies to justify $!' \circ ! = \mathrm{id}_{T'}$. □

3 Duality

Definition 3.1 (Opposite category)**.** Given a category $\mathcal{C}$, define $\mathcal{C}^{\mathrm{op}}$ to be the opposite category of $\mathcal{C}$, by setting

1. $\mathsf{ob}\ \mathcal{C}^{\mathrm{op}} := \mathsf{ob}\ \mathcal{C}$
2. $\mathsf{hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) := \mathsf{hom}_{\mathcal{C}}(Y, X)$

Exercise 3.1. Check rigorously that $\mathcal{C}^{\mathrm{op}}$ is a category.

Exercise 3.2. Check rigorously that given $f : X \cong Y$ in a category $\mathcal{C}$, then $X \cong Y$ in $\mathcal{C}^{\mathrm{op}}$.

Example 3.1. In $\mathcal{S}et^{\mathrm{op}}$, objects are still sets and for any two sets $X, Y \in \mathsf{ob}\ \mathcal{S}et$, a map $X \xrightarrow{f} Y$ exists for a function $Y \xrightarrow{f} X$ in $\mathcal{S}et$. i.e. we have $\mathsf{hom}_{\mathcal{S}et^{\mathrm{op}}}(X, Y) := \{f : Y \to X \mid f \text{ is a function}\}$.

4 Initial Objects

In Category theory, many constructions have dual constructions in the opposite category. Initial objects are an example; they can be defined in terms of terminal objects and duality.

Definition 4.1 (Initial object)**.** An initial object in a category $\mathcal{C}$ is a terminal object in the opposite category $\mathcal{C}^{\mathrm{op}}$.

The uniqueness of initial objects then follows from Lemma 2.2 (uniqueness of terminal objects). However, we can also define initial objects in their own right.

Definition 4.2 (Initial object)**.** An object I in a category $\mathcal{C}$is an initial object if for every object $Z \in \mathsf{ob}\ \mathcal{C}$, there is exactly one map $I \xrightarrow{\mathrm{i}} Z$.

It is easy to show that these two definitions are equivalent.

In $\mathcal{S}et$, the initial object is the empty set, because there exists exactly one function from the empty set to any other set: the empty function.

In $\mathcal{T}y$, the initial object is the empty type, $\perp$, for a similar reason to $\mathcal{S}et$.

In $\mathcal{P}rop$, the initial object is $\perp$, because it implies any other formula.