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Introduction to Category Theory — Paige Randall North

Lecture 3 - *July 2, 2025*

1 Products

Definition 1.1 (Product)**.** For any $X, Y \in \mathsf{ob}\ \mathcal{C}$, the product of A, B consists of

1. an object $A \times B \in \mathsf{ob}\ \mathcal{C}$
2. a pair of maps $\pi_A : A \times B \longrightarrow A$ and $\pi_B : A \times B \longrightarrow B$

such that for any $Z \in \mathsf{ob}\ \mathcal{C}$ and maps $\mathfrak{z}_A$, $\mathfrak{z}_B$ as in the diagram below, there exists a unique map $u_Z : Z \longrightarrow A \times B$ making the diagram commute.

$$\begin{tikzcd} & Z \arrow[dl, "\mathfrak{z}_A"'] \arrow[d, dashed, "u_Z"] \arrow[dr, "\mathfrak{z}_B"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

Example 1.1. Consider the following categories and their product elements.

- In $\mathscr{Set}$, products are the cartesian products $A \times B \coloneqq \{(a, b) \mid a \in A, b \in B\}$, the π functions are given by $\pi_A \coloneqq \lambda(a, b).a$ $\pi_B \coloneqq \lambda(a, b).b$ and . $\mathcal{U}_Z \coloneqq \lambda z.(\mathfrak{z}_A z, \mathfrak{z}_B z)$
- In $\mathscr{Prop}$, products are conjunctions, where $P, Q \vdash P \wedge Q$ and projections exist because $P \wedge Q \rightarrow P$ and $P \wedge Q \rightarrow Q$.

Exercise 1.1. Show that the product $A \times B$ of $A, B \in \mathsf{ob}\ \mathcal{C}$ is unique up to unique isomorphism.

Definition 1.2 (Product (Alternative))**.** Consider $A, B \in \mathsf{ob}\,\mathcal{C}$. Define $\mathcal{C}_{A,B}$ to have objects of the form:

$$\begin{tikzcd}
& Z \arrow[dl, "z_A"'] \arrow[dr, "x_B"] & \\
A & & B
\end{tikzcd}$$

(this is called a "span") and maps given by:

$$\begin{tikzcd}
& Z \arrow[dl, "z_A"'] \arrow[dr, "x_B"] & \\
A & & B \\
& Y \arrow[ul, "y_A"] \arrow[ur, "y_B"'] \arrow[uu, "f"] &
\end{tikzcd}$$

Where the map f takes spans "rooted" at A and B to spans rooted at A and B. Define the product of $A, B \in \mathsf{ob}\,\mathcal{C}$, written $A \times B$, to refer to the terminal object in $\mathcal{C}_{A,B}$.

Exercise 1.2. Show that $\mathcal{C}_{A,B}$ definition forms a category and that the terminal object of this category is equivalent to the original definition of a product.

Hint:

1. Identities are given as above when Y is Z and f is id_Z.
2. Composition is inherited from $f \circ g$ in $\mathcal{C}$in the following:

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] \arrow[d, "f"] & \\
A & Y \arrow[l] \arrow[r] \arrow[d, "g"] & B \\
& Z \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

1.1 More alternative definitions

This subsection introduces more definitions of products and terminal objects in a style that is commonly used by category theorists when carrying out computations.

Definition 1.3 (Product (Alternative))**.** Consider $A, B \in \mathsf{ob}\ \mathscr{C}$. The product of A, B is an object $A \times B \in \mathsf{ob}\ \mathscr{C}$ with the property $\mathsf{hom}_C(Z, A \times B) \cong \mathsf{hom}_C(Z, A) \times \mathsf{hom}_C(Z, B)$ natural[1] in Z.

Note: The right-hand-side $\times$ is in $\mathscr{S}et$ specifically, while the left-hand-side $\times$ indicates the product in $\mathscr{C}$ generally.

Note: This reformulates the product as a statement about related structures of maps.

Exercise 1.3. Show that this definition is, in fact, equivalent to one of the previous two definitions. *Hint:* Consider what happens for these definitions when taking the homset of maps from an object to itself.

Definition 1.4 (Alternative Terminal Object Using hom)**.** The terminal object of $\mathscr{C}$ is an object T such that $\mathsf{hom}(Z, T) \cong \{\top\}$.

2 Coproducts

Definition 2.1 (Coproduct)**.** For any $X, Y \in \mathsf{ob}\ \mathscr{C}$, the coproduct of A, B consists of

1. an object $A + B$
2. a pair of maps $\iota_A : A \to A + B$ and $\iota_B : B \to A + B$

such that for any $Z \in$ and maps $\mathfrak{z}_A, \mathfrak{z}_B$ as in the diagram below, there exists a unique map $u_Z : A + B \longrightarrow Z$ making the diagram commute.

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "\mathfrak{z}_A"] \arrow[r, "\iota_A"'] & A + B \arrow[u, dashed, "u_Z"] & B \arrow[ul, "\mathfrak{z}_B"'] \arrow[l, "\iota_B"]
\end{tikzcd}$$

Example 2.1. Consider the following examples.

- $\mathscr{S}et$: the coproduct is defined as the disjoint union $A \uplus B$.
- $\mathscr{P}rop$: the coproduct is defined as the disjunction $P \vee Q$. Thus we can say that conjunction and disjunction are *dual*.

[1]Naturality will be defined later on, the condition is included here to have a correct definition.

3 The syntactic category of the simply-typed λ-calculus

Definition 3.1. For a simply-typed λ-calculus $\mathbb{T}$, define the syntactic category $\mathcal{C}(\mathbb{T})$ by

- objects: types
- maps: derivable judgments, $x : S \vdash t : T$, quotiented by α-equivalence
- identity: $x : S \vdash x : S$
- composition: $x : S \vdash t : T$ and $y : T \vdash u : U$ derives $x : S \vdash u[t/y] : U$.

Exercise 3.1. Show that the unitality and associativity conditions hold for the above definition.

3.1 Products in STλC

Formation

$$\frac{A\ type \qquad B\ type}{A \times B\ type}$$

Introduction (or consider the case with $z : Z \rightsquigarrow \Gamma$)

$$\frac{z : Z \vdash a : A \qquad z : Z \vdash b : B}{z : Z \vdash (a, b) : A \times B}$$

Elimination

$$\frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_A p : A}$$

$$\frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_B p : B}$$

What is u_Z now? It is the introduction rule. The β and η rules say the triangles commute uniquely with unique u_Z.