



Introduction to Category Theory — Paige Randall North

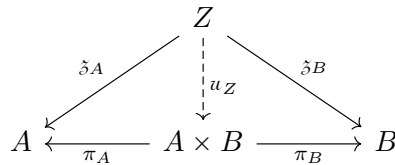
Lecture 3 - July 2, 2025

1 Products

Definition 1.1 (Product). For any $X, Y \in \text{ob } \mathcal{C}$, the product of A, B consists of

1. an object $A \times B \in \text{ob } \mathcal{C}$
2. a pair of maps $\pi_A : A \times B \longrightarrow A$ and $\pi_B : A \times B \longrightarrow B$

such that for any $Z \in \text{ob } \mathcal{C}$ and maps δ_A, δ_B as in the diagram below, there exists a unique map $u_Z : Z \longrightarrow A \times B$ making the diagram commute.

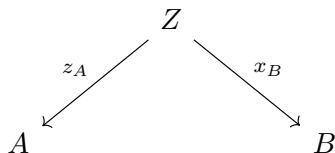


Example 1.1. Consider the following categories and their product elements.

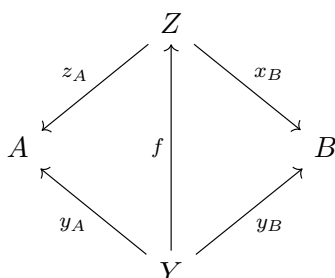
- In $\mathcal{Set}$, products are the cartesian products $A \times B := \{(a, b) \mid a \in A, b \in B\}$, the π functions are given by $\pi_A := \lambda(a, b).a$, $\pi_B := \lambda(a, b).b$ and $u_Z := \lambda z.(\delta_A z, \delta_B z)$
- In $\mathcal{Prop}$, products are conjunctions, where $P, Q \vdash P \wedge Q$ and projections exist because $P \wedge Q \rightarrow P$ and $P \wedge Q \rightarrow Q$.

Exercise 1.1. Show that the product $A \times B$ of $A, B \in \text{ob } \mathcal{C}$ is unique up to unique isomorphism.

Definition 1.2 (Product (Alternative)). Consider $A, B \in \text{ob } \mathcal{C}$. Define $\mathcal{C}_{A,B}$ to have objects of the form:



(this is called a "span") and maps given by:

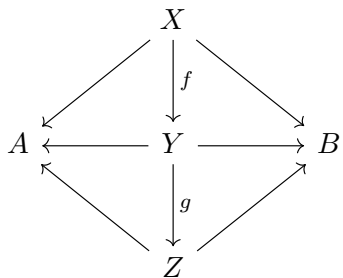


Where the map f takes spans "rooted" at A and B to spans rooted at A and B . Define the product of $A, B \in \text{ob } \mathcal{C}$, written $A \times B$, to refer to the terminal object in $\mathcal{C}_{A,B}$.

Exercise 1.2. Show that $\mathcal{C}_{A,B}$ definition forms a category and that the terminal object of this category is equivalent to the original definition of a product.

Hint:

1. Identities are given as above when Y is Z and f is id_Z .
2. Composition is inherited from $f \circ g$ in $\mathcal{C}$ in the following:



1.1 More alternative definitions

This subsection introduces more definitions of products and terminal objects in a style that is commonly used by category theorists when carrying out computations.

Definition 1.3 (Product (Alternative)). Consider $A, B \in \text{ob } \mathcal{C}$. The product of A, B is an object $A \times B \in \text{ob } \mathcal{C}$ with the property $\text{hom}_{\mathcal{C}}(Z, A \times B) \cong \text{hom}_{\mathcal{C}}(Z, A) \times \text{hom}_{\mathcal{C}}(Z, B)$ natural¹ in Z .

Note: The right-hand-side $\times$ is in Set specifically, while the left-hand-side $\times$ indicates the product in $\mathcal{C}$ generally.

Note: This reformulates the product as a statement about related structures of maps.

Exercise 1.3. Show that this definition is, in fact, equivalent to one of the previous two definitions. *Hint:* Consider what happens for these definitions when taking the homset of maps from an object to itself.

Definition 1.4 (Alternative Terminal Object Using hom). The terminal object of $\mathcal{C}$ is an object T such that $\text{hom}(Z, T) \cong \{\top\}$.

2 Coproducts

Definition 2.1 (Coproduct). For any $X, Y \in \text{ob } \mathcal{C}$, the coproduct of A, B consists of

1. an object $A + B$
2. a pair of maps $\iota_A : A \rightarrow A + B$ and $\iota_B : B \rightarrow A + B$

such that for any $Z \in$ and maps δ_A, δ_B as in the diagram below, there exists a unique map $u_Z : A + B \rightarrow Z$ making the diagram commute.

$$\begin{array}{ccccc}
 & & Z & & \\
 & \nearrow \delta_A & \uparrow u_Z & \nwarrow \delta_B & \\
 A & \xrightarrow{\iota_A} & A + B & \xleftarrow{\iota_B} & B
 \end{array}$$

Example 2.1. Consider the following examples.

- *Set*: the coproduct is defined as the disjoint union $A \uplus B$.
- *Prop*: the coproduct is defined as the disjunction $P \vee Q$. Thus we can say that conjunction and disjunction are *dual*.

¹Naturality will be defined later on, the condition is included here to have a correct definition.

3 The syntactic category of the simply-typed λ -calculus

Definition 3.1. For a simply-typed λ -calculus $\mathbb{T}$, define the syntactic category $\mathcal{C}(\mathbb{T})$ by

- objects: types
- maps: derivable judgments, $x : S \vdash t : T$, quotiented by α -equivalence
- identity: $x : S \vdash x : S$
- composition: $x : S \vdash t : T$ and $y : T \vdash u : U$ derives $x : S \vdash u[t/y] : U$.

Exercise 3.1. Show that the unitality and associativity conditions hold for the above definition.

3.1 Products in ST λ C

Formation

$$\frac{A \text{ type} \quad B \text{ type}}{A \times B \text{ type}}$$

Introduction (or consider the case with $z : Z \rightsquigarrow \Gamma$)

$$\frac{z : Z \vdash a : A \quad z : Z \vdash b : B}{z : Z \vdash (a, b) : A \times B}$$

Elimination

$$\frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_A p : A}$$

$$\frac{z : Z \vdash p : A \times B}{z : Z \vdash \pi_B p : B}$$

What is u_Z now? It is the introduction rule. The β and η rules say the triangles commute uniquely with unique u_Z .