Introduction to Category Theory — Paige Randall North

Lecture 4 - *July 3, 2025*

1 Functors

Definition 1.1. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following:

- a function $\mathsf{ob}\ F : \mathsf{ob}\ \mathcal{C} \to \mathsf{ob}\ \mathcal{D}$
- functions $F_{X,Y} : \mathsf{hom}_{\mathcal{C}}(X, Y) \to \mathsf{hom}_{\mathcal{D}}(FX, FY)$ for all $X, Y \in \mathcal{C}$

such that[1]

- $F_{X,X}\,\mathrm{id}_X = \mathrm{id}_{FX}$ for all $X \in \mathsf{ob}\ \mathcal{C}$
- $F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)$ for any $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$

Exercise 1.1. Functors preserve isomorphisms. Namely, for any functor $F : \mathcal{C} \to \mathcal{D}$, and isomorphism $f : X \cong Y$ in $\mathcal{C}$, $Ff : FX \to FY$ is also an isomorphism.

$$FX \underset{Ff^{-1}}{\overset{Ff}{\rightleftarrows}} FY$$

Example 1.1 (Identity functor)**.** Define the identity functor on $\mathcal{C}$ as follows:

- $\mathsf{ob}\ \mathrm{Id}_{\mathcal{C}} : \mathsf{ob}\ \mathcal{C} \to \mathsf{ob}\ \mathcal{C}$
  $\mathsf{ob}\ \mathcal{C} \coloneqq \lambda x.x$
- $(\mathrm{Id}_{\mathcal{C}})_{X,Y} : \mathsf{hom}_{\mathcal{C}}(X, Y) \to \mathsf{hom}_{\mathcal{C}}(X, Y)$
  $(\mathrm{Id}_{\mathcal{C}})_{X,Y} \coloneqq \lambda f.f$

[1] Notation: We write FX and Ff for $(\mathsf{ob}\ F)(X)$ and $F_{X,Y}(f)$, respectively.

Example 1.2. (Functor composition) Given three categories, $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, and two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, consider the functor composition $G \circ F : \mathcal{C} \to \mathcal{E}$ given by

- $\mathsf{ob}\ G \circ F : \mathsf{ob}\ \mathcal{C} \to \mathsf{ob}\ \mathcal{E}$
  $\mathsf{ob}\ (G \circ F) \coloneqq \lambda X.\mathsf{ob}\ G(\mathsf{ob}\ F(X))$
- $(G \circ F)_{X,Y} : \mathsf{hom}_{\mathcal{C}}(X, Y) \to \mathsf{hom}_{\mathcal{E}}(GFX, GFY)$
  $(G \circ F)_{X,Y} \coloneqq G_{FX,FY} \circ F_{X,Y}$

These two examples (with a little work left the reader) justify forming a category of categories.

Definition 1.2. The category of categories $\mathcal{C}at$ has categories themselves as objects and functors between categories as maps.

Example 1.3. Consider $\mathbb{1} : \{\cdot_A\}$ (the <u>unit</u> category), with a single element and only the identity map.

Then for every $X \in \mathsf{ob}\ \mathcal{C}$, there is a functor:

$$[X] : \mathbb{1} \to \mathcal{C}$$

that 'picks out X'. It is defined by $\mathsf{ob}\ [X] \coloneqq \lambda A.X$, and $[X]_A \coloneqq \mathrm{id}_X$.

Example 1.4. Consider a category $\mathcal{C}$. There is a functor:

$$! : \mathcal{C} \to \mathbb{1}$$

given by

- $\mathsf{ob}\ ! \coloneqq \lambda x.A : \mathsf{ob}\ \mathcal{C} \to \mathsf{ob}\ \mathbb{1}$
- $!_{X,Y} \coloneqq \lambda f.\, \mathrm{id}_A : \mathsf{hom}_C(X, Y) \to \mathsf{hom}_{\mathbb{1}}(A, A)$

Observe that the notation in the previous example is suggestive of the fact that $\mathbb{1}$ is terminal in $\mathcal{C}at$.

Exercise 1.2. Show that $\mathbb{1}$ is terminal and that the empty category is initial in $\mathcal{C}at$.

Exercise 1.3. For categories $\mathcal{C}, \mathcal{D}$, the product category $\mathcal{C} \times \mathcal{D}$ has

- objects: $\mathsf{ob}\ (\mathcal{C} \times \mathcal{D}) \coloneqq \mathsf{ob}\ \mathcal{C} \times \mathsf{ob}\ \mathcal{D}$
- maps: $\mathsf{hom}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) \coloneqq \mathsf{hom}_{\mathcal{C}}(C, C') \times \mathsf{hom}_{\mathcal{D}}(D, D')$

The coproduct category $\mathcal{C} + \mathcal{D}$ is constructed in a similar manner.

Justify that the product and coproduct categories are in fact categories, and that they are the product and coproduct of the category $\mathcal{C}at$.

Hint: for the coproduct $\mathcal{C} + \mathcal{D}$, consider the following definition for its morphisms:

- $\mathsf{hom}_{\mathcal{C}+\mathcal{D}}(C, C') \coloneqq \mathsf{hom}_C(C, C')$
- $\mathsf{hom}_{\mathcal{C}+\mathcal{D}}(D, D') \coloneqq \mathsf{hom}_D(D, D')$
- $\mathsf{hom}_{\mathcal{C}+\mathcal{D}}(C, D) \coloneqq \emptyset$
- $\mathsf{hom}_{\mathcal{C}+\mathcal{D}}(C', D') \coloneqq \emptyset$

Example 1.5. Consider the category $\mathcal{T}y$ (which has $+$, the coproduct, and T, a terminal object). Then we have a functor $\mathcal{M}aybe : \mathcal{T}y \to \mathcal{T}y$ given by the following:

- $\mathsf{ob}\ \mathcal{M}aybe \coloneqq \lambda X.X + T \colon Ty \to Ty$
- $\mathcal{M}aybe_{X,Y} : \mathsf{hom}_{\mathcal{T}y}(X, Y) \to \mathsf{hom}_{\mathcal{T}y}(\mathcal{M}aybeX, \mathcal{M}aybeY)$ given by
  - $\mathcal{M}aybe_{X,Y}\ (f : X \to Y)\ (just\ x) \coloneqq just(fx)$
  - $\mathcal{M}aybe_{X,Y}\ (f : X \to Y)\ (nothing) \coloneqq nothing$

Exercise 1.4. Consider a category $\mathcal{C}$ with coproducts and a terminal object T. Show that we can construct a functor $\mathcal{M}aybe : \mathcal{C} \to \mathcal{C}$ with specification

- $\mathsf{ob}\ \mathcal{M}aybe \coloneqq \lambda X.X + T \colon \mathcal{C} \to \mathcal{C}$
- $\mathcal{M}aybe_{X,Y} : \mathsf{hom}_{\mathcal{T}y}(X, Y) \to \mathsf{hom}_{\mathcal{T}y}(\mathcal{M}aybeX, \mathcal{M}aybeY)$

Hint: Prove (and use) the following lemma.

Lemma 1.1. In a given category $\mathcal{C}$‘ which has products/coproducts, given two maps $X \xrightarrow{f} X'$ and $Y \xrightarrow{g} Y'$, it is possible to construct maps

- $X + Y \xrightarrow{f+g} X' + Y'$
- $X \times Y \xrightarrow{f \times g} X' \times Y'$