OREGON PROGRAMMING LANGUAGES SUMMER SCHOOL

Introduction to Logical Foundations — Brigitte Pientka

Lecture 1 - *June 23, 2025*

This lecture gives an introduction to constructive logic, focusing on Gentzen's natural deduction, a particular logical system designed to be more modular than the Hilbert-style "list of axioms." In order to understand the Curry-Howard isomorphism, we begin with exploring connections between proofs and propositions, as well as types and programs.

1 Propositions

To talk about natural deduction, we first need to define what a proposition is.

Definition 1.1 (Inductive Definition of Well-Formed Proposition)**.** A *proposition* is one of the following:[1]

- $\top$ (true) is a proposition
- if A, B are propositions, then so are $A \wedge B$, $A \vee B$, $A \supset B$

We use A, B, C to range over propositions. In BNF form, it would be defined as:

$$\text{Proposition } A, B, C \coloneqq \top \mid A \wedge B \mid A \vee B \mid A \supset B.$$

To give meaning to such propositions, let's now look at a *judgmental view* of propositions.

A quote from Per Martin-Löf: "*The idea is that the analytical, or explicative, judgements are those that become evident merely by conceptual analysis, that is, they are those whose evidence rests on conceptual analysis alone.*" [2]

[1] For simplicity of exposition, we are not including a false proposition. However, when we use metavariables A, B variables for propositions, we will not make any assumptions about it being some particular piece of syntax other than just the symbols A or B. Thus, it makes sense to say "we cannot prove $A \supset B$", even though (because we did not include a false proposition) we will be able to prove $A \supset X$ for every concrete piece of syntax X generated by our grammar.

To illustrate this idea with some further details, Martin-Löf uses an example of weather and a thermometer displaying 25 degrees in [2].

Consider the judgment "It is raining." Martin-Löf shows that the nature of this judgment depends critically on what evidence we consider:

- **Synthetic judgment**: If we treat "It is raining" as a standalone linguistic utterance, then no amount of conceptual analysis can establish its truth. To verify this judgment, we must appeal to *external evidence*—either direct observation of falling rain or some indirect evidence. The judgment is *synthetic* because its truth depends on empirical facts beyond the linguistic expression itself.
- **Analytic judgment**: However, if we consider the *complete complex* consisting of both the linguistic part ("It is raining") and the nonlinguistic part (the actual falling rain), then the judgment becomes *analytic*. Everything needed to verify the judgment is contained within this complex—no additional evidence is required.

The temperature example of having a thermometer displaying 25 degrees follows the same pattern:

- The judgment "The temperature is +25°C" is **synthetic, not analytic** when considered as a mere verbal expression. Analyzing the concepts involved cannot establish its truth.
- When we consider the judgment together with the thermometer reading +25°C, the complete complex becomes **analytic**. All necessary evidence is present within the complex itself.

For the scope of this lecture, we are mainly concerned with analytical judgements.

In general, an (analytic) judgment in the form $\boxed{A\ \texttt{wf}}$ defines when a proposition is well-formed. It is defined by a set of *inference rules*, which have the general form

$$\frac{\mathcal{J}_1 \ldots \mathcal{J}_n}{\mathcal{J}}$$

where $\mathcal{J}_1 \ldots \mathcal{J}_n$ are the *premises* and $\mathcal{J}$ is the *conclusion*.

We introduce the judgment $A\ \texttt{wf}$, saying that proposition A is "well-formed" (*i.e.*, syntactically valid), which we define by the following rules:

$$\frac{}{\top\ \texttt{wf}} \qquad\qquad \frac{A\ \texttt{wf} \qquad B\ \texttt{wf}}{A\ op\ B\ \texttt{wf}}(op \in \{\wedge, \vee, \supset\})$$

2 Natural deduction

We introduce the judgment $\boxed{A \texttt{ true}}$. To define $\boxed{A \texttt{ true}}$, we will use both

- *introduction rule*, which introduces a connective on the bottom, denoted by I.
- *elimination rule*, saying how to obtain information from a given connective, denoted by E.

The proposition $\top$ should always be true. Resultantly, the judgement $\top$ `true` is true holds unconditionally and has no premises.

$$\frac{}{\top \texttt{ true}}\top I \qquad \frac{A \texttt{ true} \quad B \texttt{ true}}{A \wedge B \texttt{ true}}\wedge I \qquad \frac{A \wedge B \texttt{ true}}{A \texttt{ true}}\wedge E_l \qquad \frac{A \wedge B \texttt{ true}}{B \texttt{ true}}\wedge E_r$$

Here are some examples of using these rules to build proofs. Here is a proof of $\top \wedge (\top \wedge \top)$, and a **sketch of proof** of B starting from $A \wedge (B \wedge C)$.

$$\frac{\frac{}{\top \texttt{ true}}\top I \qquad \frac{\frac{}{\top \texttt{ true}}\top I \quad \frac{}{\top \texttt{ true}}\top I}{\top \wedge \top \texttt{ true}}\wedge I}{\top \wedge (\top \wedge \top) \texttt{ true}}\wedge I \qquad\qquad \frac{\frac{A \wedge (B \wedge C) \texttt{ true}}{B \wedge C \texttt{ true}}\wedge E_r}{B \texttt{ true}}\wedge E_l$$

The second example, of course, is not fully complete nor valid: there's currently no rule that gave us $A \wedge (B \wedge C)$ `true` and it is currently unjustified. Instead, we're trying to do *hypothetical reasoning*, saying, if we knew $A \wedge (B \wedge C)$, then we could derive B.

2.1 Hypothetical Derivation and Reasoning

From the second example above, what we are really trying to do is, given $A \wedge (B \wedge C)$ `true` as an assumption, we aim to prove that B `true`. Written in the proposition form, we'd like to prove the *implication* $A \wedge (B \wedge C) \supset B$.

Generally speaking, we might have more than one assumption. Therefore, the *hypothetical derivations* have the form as follows:

$$\cfrac{\cfrac{}{J_1}H_1 \quad \dots \quad \cfrac{}{J_n}H_n}{\cfrac{\vdots}{J}}$$

This is to say, we can derive J given the assumptions $J_1 \dots J_n$. We don't claim that we can prove $J_1 \dots J_n$.

Substitution Principle These assumptions are unproven, and one we can build a derivation for any J_i, we can "plug" such derivations into the corresponding places in the derivation tree and eliminate usage of the assumptive version of J_i by referring to the actual derivations we built. This is also known as the *substitution principle* for hypothesis.

Implication Using notions of hypothetical judgements, we can now explain the meaning of $A \supset B$ (A implies B), as implications already internalizes hypothetical reasoning. Similarly, we introduce rules for implications:

$$\cfrac{\begin{array}{c}\cfrac{}{A\ \texttt{true}}u \\ \vdots \\ B\ \texttt{true}\end{array}}{A \supset B\ \texttt{true}} \supset I^u \qquad\qquad \cfrac{A \supset B\ \texttt{true} \qquad A\ \texttt{true}}{B\ \texttt{true}} \supset E$$

Scoping The label u denotes the assumption $\boxed{A\ \texttt{true}}$. Referring to this label again at the bottom of the inference rule with the name clarifies the *scope* of assumption u: We make an assumption u to prove $\boxed{\text{B}\ \texttt{true}}$ only and then *discharge* it once the proof is done. In this way, the assumptions are delimited with their scopes and we cannot use them outside of such borders. This also follows Gentzen's top-down notation for natural deduction[1].

Now we can complete our previous example in the following way:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \wedge (B \wedge C)\ \texttt{true}}(u)}{B \wedge C\ \texttt{true}} \wedge E_r}{B\ \texttt{true}} \wedge E_l}{A \wedge (B \wedge C) \supset B\ \texttt{true}} \supset I^u$$

Structural Laws Assumptions have the following properties, which we call *structural laws*.

- **Weakening:** Assumptions don't have to be used, and additional unused assumptions do no harm.

  This is used, for instance, in proving $A \supset B \supset A$ $\texttt{true}$ (which we implicitly take to mean $A \supset (B \supset A)$).

- **Contraction:** Assumptions can be used as often as you like, not only once.

  This is used, for instance, in proving $A \supset (A \wedge A)$ $\texttt{true}$.

3 Natural deduction with contexts

As presented above, assumptions are somewhat odd / cumbersome to formalize. We'll rephrase our natural deduction system by adding *contexts*, to make assumptions more explicit. We define

$$\text{Context } \Gamma \coloneqq \cdot \mid \Gamma, x : A \ \texttt{true}$$

where x is any name / identifier. We think of Γ as a set of assumptions.

Our new truth judgment is now $\Gamma \vdash A$ $\texttt{true}$. Our rules turn into the following:

$$\frac{}{\Gamma \vdash \top \ \texttt{true}} \top I \qquad \frac{\Gamma \vdash A \ \texttt{true} \qquad \Gamma \vdash B \ \texttt{true}}{\Gamma \vdash A \wedge B \ \texttt{true}} \wedge I \qquad \frac{\Gamma \vdash A \wedge B \ \texttt{true}}{\Gamma \vdash A \ \texttt{true}} \wedge E_l$$

$$\frac{\Gamma \vdash A \wedge B \ \texttt{true}}{\Gamma \vdash B \ \texttt{true}} \wedge E_r$$

and our new rules for implication become

$$\frac{\Gamma, u : A \ \texttt{true} \vdash B \ \texttt{true}}{\Gamma \vdash A \supset B \ \texttt{true}} \supset I^u \qquad \frac{\Gamma \vdash A \supset B \ \texttt{true} \qquad \Gamma \vdash A \ \texttt{true}}{\Gamma \vdash B \ \texttt{true}} \supset E \qquad \frac{(u : A \ \texttt{true}) \in \Gamma}{\Gamma \vdash A \ \texttt{true}} (u)$$

Now, rather than trying to state what the properties of assumptions ought to be in words, we can write down these properties formally, and then prove that they hold, using induction:

- **Weakening**: if $\Gamma, \Gamma' \vdash A$ $\texttt{true}$, then $\Gamma, u : B \ \texttt{true}, \Gamma' \vdash A$ $\texttt{true}$
- **Contraction**: if $\Gamma, x : B \ \texttt{true}, y : B \ \texttt{true}, \Gamma' \vdash A$ $\texttt{true}$, then $\Gamma, x : B \ \texttt{true}, \Gamma' \vdash A$ $\texttt{true}$
- **Substitution**: if $\Gamma, x : A \ \texttt{true}, \Gamma' \vdash B$ $\texttt{true}$, and $\Gamma \vdash A$ $\texttt{true}$, then $\Gamma, \Gamma' \vdash B$ $\texttt{true}$

4 Local Soundness and Local Completeness

Now we turn to the question of how we know we have the right rules. We will split this into two criteria: local soundness and local completeness. These are purely proof-theoretic criteria, rather than depending on some chosen specific semantics for the logic. We aim to choose a proper combination of introduction and elimination rules, so that they will meet certain conditions:

- **Soundness:** They should not allow us to deduce unintended new truths
- **Completeness:** They should be strong enough to seek out all the information contained in the given connectives.

To give a more relevant picture:

- **Local Soundness:** If we introduce a connective and then immediately eliminate it, we should be able to delete this detour completely and obtain a direct derivation ending up with the *exact same conclusion*.

  If this property does not hold, then the elimination rules are too strong. (They allow us to derive more information than it should be).

- **Local Completeness:** If we start with a compound formula (built on connectives), we can eliminate a connective in a way that it still keeps sufficient information to reconstruct the connective by an introduction rule. The key idea is that we should always be able to rebuild (using introduction) what we started with using just the pieces we get from elimination.

  If this property does not hold, then the elimination rules are too weak: they do not allow us to obtain everything we should be able to and we are losing information in this process.

Both of these notions fit into the idea of transforming a proof of a proposition into an intuitively "simpler" proof of that proposition. Using $\wedge$ as an example:

- Suppose the proposition P does not contain $\wedge$. You could still use $\wedge I$ and $\wedge E$ in its proof, but it would be simpler if a proof of P only involves the connectives in P.

  Local soundness shows how to simplify a proof in this way, when $\wedge I$ and $\wedge E$ occur right next to each other (rather than maybe occurring far away from each other in the proof—this is what "local" is referring to).

- Suppose P has the form $P_1 \wedge P_2$. It would be simplest to prove it using $\wedge I$ as the last rule, though of course, it is not necessary.

  Local completeness shows how to transform any proof of $P_1 \wedge P_2$ into one whose final (bottom-most) proof rule is $\wedge I$.

4.1 Local soundness

Local soundness captures the intuitive idea that the combination of introduction rules and elimination rules is not *too strong*: that is, they don't allow us to infer more than we already know. Concretely, if we apply an elimination rule immediately to an introduction rule, we should be able to simplify it.

For example, if we start with the left hand side, we can simplify it to the right:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma \vdash A \ \texttt{true}} \qquad \Gamma \vdash B \ \texttt{true}}{\Gamma \vdash A \wedge B \ \texttt{true}} \wedge I}{\Gamma \vdash A \ \texttt{true}} \wedge E_l \qquad\qquad \dfrac{\mathcal{D}}{\Gamma \vdash A \ \texttt{true}}$$

and symmetrically for the other elimination rule, $\wedge E_r$.

For implication, given the left, we can make the right using substitution:

$$\dfrac{\dfrac{\Gamma, u : A \ \texttt{true} \vdash B \ \texttt{true}}{\Gamma \vdash A \supset B \ \texttt{true}} \wedge I^u \qquad \Gamma \vdash A \ \texttt{true}}{\Gamma \vdash B \ \texttt{true}} \supset E \qquad\qquad \dfrac{\cdots}{\Gamma \vdash B \ \texttt{true}}$$

The trivially true proposition $\top$ has no elimination rules, so local soundness for $\top$ is trivial.

You can see that it is unnecessary to first introduce a conjunction and then immediately eliminate it, since we can discover a more direct proof already. Therefore, it's possible that many different proofs for a same proposition can be collapsed to a more direct proof without the detour. Such proof collapsing process is referred as normalisation (or trying to find the normal form of a proof).

4.2 Local Completeness

Local completeness captures the intuitive idea that the combination of intro and elim rules is *strong enough*. Concretely, we would like to know that whenever we can prove some connective, then we could just as well prove it using the introduction rule as our last rule. (This shows that the intro rule is strong enough to prove anything that can be proved.)

For $\wedge$, if we have the left, we can expand it to the right thing:

$$\frac{\mathcal{D}}{\Gamma \vdash A \wedge B \ \texttt{true}} \qquad\qquad \dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma \vdash A \wedge B \ \texttt{true}}}{\Gamma \vdash A \ \texttt{true}} \wedge E_l \qquad \dfrac{\dfrac{\mathcal{D}}{\Gamma \vdash A \wedge B \ \texttt{true}}}{\Gamma \vdash B \ \texttt{true}} \wedge E_r}{\Gamma \vdash A \wedge B \ \texttt{true}} \wedge I$$

and for $\supset$, given the first, we can make the second using weakening:

$$\frac{\mathcal{D}}{\Gamma \vdash A \supset B \ \texttt{true}} \qquad\qquad \dfrac{\dfrac{\dfrac{\cdots}{\Gamma, x : A \ \texttt{true} \vdash A \supset B \ \texttt{true}} \qquad \dfrac{}{\Gamma, x : A \ \texttt{true} \vdash A \ \texttt{true}}}{\Gamma, x : A \ \texttt{true} \vdash B \ \texttt{true}} \supset E}{\Gamma \vdash A \supset B \ \texttt{true}} \supset I^x$$

References

[1] Gerhard Gentzen. “Untersuchungen über das logische Schließen. I”. In: *Mathematische zeitschrift* 39.1 (1935), pp. 176–210.

[2] Per Martin-Löf. “Analytic and synthetic judgements in type theory”. In: *Kant and contemporary epistemology*. Springer, 1994, pp. 87–99.