OREGON PROGRAMMING LANGUAGES SUMMER SCHOOL

Introduction to Logical Foundations — Brigitte Pientka

Lecture 2 - Curry-Howard Isomorphism *June 23, 2025*

The Curry-Howard Isomorphism is a relation between:

propositions	$\leftrightarrow$	types
proofs	$\leftrightarrow$	programs
natural deduction calculus	$\leftrightarrow$	(extended) λ-calculus

What does this look like?

propositions	**types**	**programs**
$A \wedge B$	$A \times B$ "cross product"	$\langle M, N \rangle$, $\mathsf{fst}\ M, \mathsf{snd}\ N$
$A \supset B$	$A \to B$	$\lambda x : A.M,\ \ M\ \ N$
$\top$	unit (1)	$()$
$A \vee B$	$A + B$ "disjoint sum"	$\mathsf{inl}\ M, \mathsf{inr}\ M, \mathsf{case}$
$\forall x : \tau.A$	$\prod x : \tau.A$	indexed type / dependent type (Π)
$\exists : \tau.A$	$\sum x : \tau.A$	Indexed (Dependent) Type ($\sum$)
$\forall : o.A$	$\forall \alpha.A$	Polymorphic Type

Why is this important?

1. Logic is a guide to programming language design.
2. It is the basis of type theory.
3. It gives us a way of proving consistency of a logic through 'looking' at programs.

Exercise. $(A \vee B) \supset (A \wedge B)$

Given some assumptions in the context Γ,

$\Gamma \vdash M : A$ means that:

- M is a proof term/witness corresponding to the proposition 'A true'
- M is a program of type A.

By design, $\Gamma \vdash A$ true if and only if $\Gamma \vdash M : A$. Additionally, these have an identical structure of derivation.

We exhibit/reveal the actual structure of M while preserving the type.

proof reduction	$\leftrightarrow$	program reduction
normalizing proofs	$\leftrightarrow$	normalizing programs

1 Conjunction

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \wedge B} \wedge I$$

$$\frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \mathsf{fst}\ M : A} \wedge E_l$$

$$\frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \mathsf{snd}\ M : A} \wedge E_r$$

1.1 Local Soundness

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \vdash M : A} \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \wedge B} \wedge I}{\Gamma \vdash \mathsf{fst}\ \langle M, N \rangle : A} \wedge E_l \implies \overset{\mathcal{D}}{\Gamma \vdash M : A}$$

$$\mathsf{fst}\ \langle M, N \rangle \implies M$$

$$\mathsf{snd}\ \langle M, N \rangle \implies N$$

Subject Reduction (Type Preservation):

If $\Gamma \vdash M : A$ and $M \Rightarrow M'$, then $\Gamma \vdash M' : A$

Reductions could be chained

$$M_1 \Rightarrow M_2 \cdots \Rightarrow M_n$$

If no reductions are possible, M_n is called a normal form.

1.2 Local Completeness

$$\Gamma \vdash M : A \wedge B \Longrightarrow \dfrac{\dfrac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \mathsf{fst}\ M : A} \wedge E_\ell \quad \dfrac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \mathsf{snd}\ M : B} \wedge E_r}{\Gamma \vdash \langle \mathsf{fst}\ M, \mathsf{snd}\ M \rangle : A \wedge B} \wedge I$$

2 Implications

$$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \supset B} \supset I^x \qquad \dfrac{\Gamma \vdash M : A \supset B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \supset E$$

2.1 Local Soundness

$$\dfrac{\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \supset B} \supset I^x \quad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x : A.M)\ N : B} \supset E \Longrightarrow \Gamma \vdash [N/x]\ M : B \qquad \text{(substitution lemma)}$$

$$(\lambda x : A.M)\ N \Longrightarrow [N/x]\ M \qquad (\beta)$$

2.2 Local Completeness

$$\Gamma \vdash M : A \supset B \Longrightarrow \dfrac{\dfrac{\dfrac{\Gamma \vdash M : A \supset B}{\Gamma, x : A \vdash M : A \supset B} \quad \dfrac{}{\Gamma, x : A \vdash x : A}}{\Gamma, x : A \vdash M\ x : B} ?}{\Gamma \vdash (\lambda x : A.M)\ x : A \supset B} \supset I^x$$

$$M : A \supset B \Longrightarrow \lambda x : A.M\ x \tag{η}$$

$$\dfrac{\dfrac{x : A \wedge A \vdash x : A \wedge A}{x : A \wedge A \vdash \mathsf{fst}\ x : A} \wedge E_l}{\vdash \lambda x : A \wedge A.\mathsf{fst}\ x : A \wedge A \supset A} \supset I^x$$

$$\dfrac{}{\vdash \lambda x : A \wedge A.\mathsf{fst}\ x : A \wedge A \supset A} \supset I^x$$

3 Disjunction

$$\dfrac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{inl}\ M : A \vee B} \vee I_l \qquad \dfrac{\Gamma \vdash M : B}{\Gamma \vdash \mathsf{inr}\ M : A \vee B} \vee I_r$$

$$\dfrac{\Gamma \vdash M : A \vee B \quad \Gamma, x : A \vdash N_1 : C \quad \Gamma, x : B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}\ x \to N_1 \mid \mathsf{inr}\ x \to N_2 : C} \vee E$$

Note that there is another potential definition for disjunction:

$$\dfrac{\Gamma \vdash A \vee B \quad \Gamma \vdash A \supset C \quad \Gamma \vdash B \supset C}{\Gamma \vdash C}$$

This makes sense, but it is not good, as it is defined with respect to another symbol.

3.1 Local Soundness

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash M : A\end{array}}{\Gamma \vdash \mathsf{inl}\ M : A \vee B} \vee I_l \quad \begin{array}{c}\mathcal{E}\\ \Gamma, x : A \vdash N_1 : C \quad \Gamma, x : B \vdash N_2 : C\end{array}}{\Gamma \vdash \mathsf{case}\ (\mathsf{inl}\ M)\ \mathsf{of}\ \mathsf{inl}\ x \to N_1 \mid \mathsf{inr}\ x \to N_2 : C} \vee E$$

By substitution lemma with $\mathcal{D}$ and $\mathcal{E}$