Introduction to Logical Foundations — Brigitte Pientka

Lecture 3 - A Judgmental Reconstruction of Modal Logic (S4) *June 25, 2025*

Slogan: Truth is living in the moment - here and now; validity is living forever and everywhere.

1 Modal logic

Definition (Necessity Modality ($\Box A$)). For proposition A, $\Box A$ can be read as "A is necessarily true".

Definition (Validity).

- If $\cdot \vdash A$ true then A valid.
- If A valid then $\Gamma \vdash A$ true. (weakening)

2 Judgment anatomy

A judgment looks like

$$\Delta; \Gamma \vdash A \ \mathsf{true}$$

where

- Δ contains "global" assumptions which are valid "forever".
  Looks like $x_1 : A_1$ valid, $\ldots$, $x_n : A_n$ valid
- Γ contains "local" assumptions which are valid "here and now".
  Looks like $y_1 : B_1$ true, $\ldots$, $y_m : B_m$ true

3 Hypothesis rules

$$\frac{x : A \text{ true} \in \Gamma}{\Delta; \Gamma \vdash A \text{ true}} \qquad \frac{x : A \text{ valid} \in \Delta}{\Delta; \Gamma \vdash A \text{ true}}$$

$$\frac{\Delta; \cdot \vdash A \text{ true}}{\Delta; \Gamma \vdash \Box A \text{ true}} \Box I \qquad \frac{\Delta; \Gamma \vdash \Box A \text{ true} \quad \Delta, u : A \text{ valid}; \Gamma \vdash C \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} \Box E$$

3.1 Old and bad

Here's an old version of the box introduction rule that doesn't work so well, actually:

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma, \Gamma' \vdash \Box A}$$

This is from Prawitx '65. Unfortunately, it is not locally complete.

4 Local soundness / completeness

We want to show that both $\Box I$ and $\Box E^u$ are locally sound and complete.

$$\frac{\dfrac{\overset{\mathcal{D}}{\Delta; \cdot \vdash A \text{ true}}}{\Delta; \Gamma \vdash \Box A \text{ true}} \Box I \qquad \overset{\mathcal{E}}{\Delta, u : A \text{ valid}; \Gamma \vdash C \text{ true}}}{\Delta; \Gamma \vdash C \text{ true}} \Box E^u \implies \Delta; \Gamma \vdash C \text{ true}$$

Lemma (Substitution Lemma)**.** *If* $\Delta; \Gamma, A$ *true* $\vdash C$ *true and* $\Delta; \Gamma \vdash A$ *true, then* $\Delta; \Gamma \vdash C$ *true.*

Lemma (Modal Substitution)**.** *If* $(\Delta, y : A$ *valid*$); \Gamma \vdash C$ *true and* $\Delta; \cdot \vdash A$ *true, then* $\Delta; \Gamma \vdash C$ *true*

5 Properties

- **Reflexivity:** $\Box A \supset A$
- **Distributivity:** $\Box(A \supset B) \supset \Box A \supset \Box B$
- **Transitivity:** $\Box A \supset \Box\Box A$

5.1 Reflexivity proof

$$\dfrac{\dfrac{}{\cdot; x : \Box A \text{ true}}\, x \qquad \dfrac{}{\cdot; y : A \text{ valid}; x : \Box A \text{ true} \vdash A \text{ true}}\, y}{\dfrac{\cdot; x : \Box A \text{ true} \vdash A \text{ true}}{\cdot; \cdot \vdash \Box A \supset A \text{ true}} \supset I^x}$$

5.2 Transitivity proof

6 Actual Modal Logic Grammar

Terms $M := x \mid \lambda x : A.M \mid M\ N \mid \langle M, N \rangle \mid \text{fst } M \mid \text{snd } M \mid u \mid \text{box } M$

where u is for valid assumptions.

7 Real rules

$$\dfrac{y : A \text{ true} \in \Gamma}{\Delta; \Gamma \vdash y : A} \qquad \dfrac{u : A \text{ valid} \in \Delta}{\Delta; \Gamma \vdash u : A}$$

$$\dfrac{\Delta; \cdot \vdash M : A}{\Delta; \Gamma \vdash \text{box } M : \Box A} \Box I \qquad \dfrac{\Delta; \Gamma \vdash M : \Box A \quad (\Delta; u : A \text{ valid}); \Gamma \vdash N : C}{\Delta; \Gamma \vdash \text{let box } u = M \text{ in } N : C} \Box E$$

7.1 Completeness of elim

$$\dfrac{\dfrac{TODO}{\Delta; \Gamma \vdash M : \Box A} \quad \begin{matrix}\mathcal{E} \\ (\Delta; u : A \text{ valid}); \Gamma \vdash N : C\end{matrix}}{\Delta; \Gamma \vdash \text{let box } u = M \text{ in } N : C} \Box E$$

7.2 Substitution

We now have a need to differentiate between different kinds of substitution:

$$[M/x]\ N$$

$$[M/x]\ (\mathsf{box}\ N) = \mathsf{box}\ N$$

$$[\![M/u]\!]\ (\mathsf{box}\ N) = \mathsf{box}\ ([\![M/u]\!]\ N)$$

7.3 Example function

Here is a definition of a function that takes in an int n and a vector of booleans, and outputs the nth boolean in the vector.

$$\mathsf{nth} : \texttt{int} \to \Box(\texttt{bool\_vec} \to \texttt{bool})$$

$$\mathsf{nth}\ 0 = \Box(\mathsf{fun}\ v \to \mathsf{hd}\ v)$$

$$\mathsf{nth}\ (s\ n) = (\mathsf{let\ box}\ r = \mathsf{nth}\ n\ \mathsf{in\ box}\ (\mathsf{fun}\ v \to r(\mathsf{tl}\ v)))$$

7.4 Updated definition

Definition (Validity). Our definition of validity has changed for our updated system: If $\Psi \vdash A$ true then A valid wrt Ψ

7.5 Contextual types

If we have a function $\mathsf{fun}\ x \to \bigcirc + x$, the type of the function is $\texttt{int}\to\texttt{int}$, and the type of the hole, interestingly, is $x : \texttt{int} \Vdash \texttt{int}$, which means that we are allowed to use x and not just `int` literals.