OREGON PROGRAMMING LANGUAGES SUMMER SCHOOL

# *Introduction to Logical Foundations* — Brigitte Pientka

Lecture 5 (Substructural Logic) - *June 28, 2025*

**Slogan of today:** Truth is ephemeral.

For this lecture, we talked about substructural logics. These are the kinds of logics we get when we remove some of the structural rules:

- Weakening (W)
  - Additional assumptions don't harm.
- Contraction (C)
  - We can reuse assumptions.

We get several logics by admitting/excluding these rules:

| Logic | Structural rules |
|---|---|
| Intuitionistic | W, C |
| Linear | $\emptyset$ |
| Affine | W |
| Relevant | C |

(Intuitionistic logic is the one introduced in lectures 1 and 2.)

Now, why would we want to remove structural rules? Take the following situation: suppose we start with the following context:

$$\Gamma := t : \$1 \supset \textbf{tea}, c : \$1 \wedge \$1 \supset \textbf{coffee}.$$

In English, tea costs \$1 and coffee costs \$2. We will now show that $\Gamma \vdash \$1 \supset \textbf{coffee}$. This should be, intuitively untrue. Yet, with intuitionistic logic, we can end up in this state, as can be proven below.

$$\cfrac{\cfrac{}{\Gamma \vdash \$1 \wedge \$1 \supset \textbf{coffee}}\,\text{c} \qquad \cfrac{\cfrac{}{\Gamma, x : \$1 \vdash \$1}\,\text{x} \quad \cfrac{}{\Gamma, x : \$1 \vdash \$1}\,\text{x}}{\Gamma, x : \$1 \vdash \$1 \wedge \$1}\,\wedge I}{\cfrac{\Gamma, x : \$1 \vdash \textbf{coffee}}{\Gamma \vdash \$1 \supset \textbf{coffee}}\,\supset I}\,\supset E$$

Uh oh! We copied money somehow. What's the answer here? One is appealing to *linear logic*.

# 1 Linear Logic

The intuition behind linear logic is that we have to use all of the assumptions as many times as they appear in our context. To do that, we have to effectively disable weakening and contraction. We have to change our rules. We begin with the variable rule.

## 1.1 Variables

We still have judgements of the form

$$\Delta \vdash A \text{ true.}$$

Recall that the rule in the normal intuitionistic logic goes as follows.

$$\cfrac{}{\Gamma, x : A \text{ true} \vdash A \text{ true.}}\, x$$

Here, we're allowed to keep the full context $\Gamma$. In linear logic, we do not have this luxury. Below is the rule in linear logic.

$$\cfrac{}{x : A \text{ true} \vdash A \text{ true.}}\, x$$

Why is it like this? Well, in linear logic, we want to ensure that we've used every variable in our context. This means that, by the time we've reached the point that we are using our variable rules, we better have used everything besides the variable up!

## 1.2 Implications $\multimap$:

In the linear setting, we use a different notation for implication: $A \multimap B$. Our rules are as follows.

$$\cfrac{\Delta, x : A \text{ true} \vdash B \text{ true}}{\Delta \vdash A \multimap B \text{ true}}\, \multimap I^x$$

$$\frac{\Delta_1 \vdash A \multimap B \text{ true} \qquad \Delta_2 \vdash A \text{ true}}{\Delta_1; \Delta_2 \vdash B \text{ true}} \multimap E$$

What is this semicolon? Intuitively, it's just a merging of contexts. More formally:

### 1.3 Defining Merge: $\Delta_1; \Delta_2$

$$(\Delta, x : A \text{ true}); \Delta_2 := (\Delta_1; \Delta_2), x : A \text{ true if } x \notin \Delta_2$$
$$\Delta_1; (\Delta_2, x : A \text{ true}) := (\Delta_1; \Delta_2), x : A \text{ true} \qquad \text{if } x \notin \Delta_1$$
$$\Delta; \bullet := \Delta$$
$$\bullet; \Delta := \Delta$$

Note that $\Delta_1; \Delta_2$ is not defined if $x$ is in both contexts! In the previous lectures (on modal logics), ";" was actually a symbol in our language and not a metatheoretic operation as here!

So, what is the intuition behind this rule? It looks almost like the original implication ($\supset$) that we've looked at so far. The key difference is that we split the context into a disjoint, and send one to the left and one to the right. Why? We cannot reuse propositions.

**Example 1.1.**

$$\dfrac{\dfrac{\dfrac{\dfrac{}{f : A \multimap B \vdash A \multimap B}\, f \qquad \dfrac{}{x : A \vdash A}\, x}{x : A, f : A \multimap B \vdash B} \multimap E}{x : A \vdash (A \multimap B) \multimap B} \multimap I}{\bullet \vdash A \multimap (A \multimap B) \multimap B} \multimap I$$

### 1.4 Conjunction $\otimes$:

We need to rethink conjunction. In fact, we need two forms of conjunction. First, we have the "eager pair", or "simultaneous conjunction".

$$\frac{\Delta_1 \vdash A \text{ true} \qquad \Delta_2 \vdash B \text{ true}}{\Delta_1; \Delta_2 \vdash A \otimes B \text{ true}} \otimes I$$

**Remark 1.2.** The symbol $\otimes$ is read as "tensor".

$$\frac{\Delta_1 \vdash A \otimes B \text{ true} \qquad \Delta_2, x : A \text{ true}, y : B \text{ true} \vdash C \text{ true}}{\Delta_1; \Delta_2 \vdash C \text{ true}} \otimes E$$

How do we read these rules? The introduction rule says "I have the resource to achieve $A$ and $B$ at the same time".

### 1.5 Alternative Conjunction & (internal choice/lazy pair):

Now, we have an alternative conjunction that looks more or less like the conjunction from intuitionistic logic. We call this one the "lazy pair". For reasons that will become apparent when we discuss disjunction, this is also called the "external choice".

$$\frac{\Delta \vdash A \text{ true} \qquad \Delta \vdash B \text{ true}}{\Delta \vdash A \& B \text{ true}} \& I$$

$$\frac{\Delta \vdash A \& B \text{ true}}{\Delta \vdash A \text{ true}} \& E_l$$

$$\frac{\Delta \vdash A \& B \text{ true}}{\Delta \vdash B \text{ true}} \& E_r$$

Note that we do not split the context in the introduction rule. Why? The idea here is that by inspecting the pair $A\&B$ via elimination, we are throwing away one or the other. Thus, when we prove the type $A\&B$, we need to be prepared to show that $A$ and $B$ are true regardless of which side we picked.

### 1.6 Disjunction ⊕ (External Choice)

Disjunction requires a small change in the elimination rule. The introduction rule is exactly the same as the the one we've learned for ∨. As for elimination, recall that in the intuitionistic logic, we do not know whether the element from the conjunction we will use is on the left or the right. That meant we had to prove both simultaneously. The same idea goes here, but we have to keep in mind that we cannot reuse contexts. So, we must split/merge contexts in the elimination rule.

$$\frac{\Delta \vdash A \text{ true}}{\Delta \vdash A \oplus B \text{ true}} \& I_l$$

$$\frac{\Delta \vdash B \text{ true}}{\Delta \vdash A \oplus B \text{ true}} \& I_r$$

$$\frac{\Delta_1 \vdash A \oplus B \text{ true} \qquad \Delta_2, x : A \text{ true} \vdash C \text{ true} \qquad \Delta_2, y : B \text{ true} \vdash C \text{ true}}{\Delta_1 ; \Delta_2 \vdash C \text{ true}} \oplus E^{y,x}$$

There is one more disjunction ⅋ called "par", but we do not cover it in this lecture.

## 2 Local Soundness and Completeness

We leave it as an exercise for the reader to prove that the rules introduced so far are locally sound and complete. To be able to solve this exercise, one needs the following version of the substitution lemma (which we leave as an exercise as well):

**Lemma 2.1.** *If* $\Delta_1 \vdash A$ true *and* $\Delta_2, x : A$ true $\vdash C$ true*, then* $\Delta_1; \Delta_2 \vdash C$ true*.*

## 3 Putting the Logics Together

Some people have concerned themselves with combining these two logics. Basically, linear logic is weak, but it has some strong guarantees about resource usage. Unrestricted logic is strong, but it lacks those guarantees. Can we make these work together?

One idea is to have disjoint contexts. Let $\Gamma$ be an unrestricted context (admitting weakening and contraction), and $\Delta$ be a linear context. Then we have the judgment:

$$\Gamma \mid \Delta \vdash A \text{ true}$$

We preserve all the same rules, but add $\Gamma$ to the context. The two contexts interact using the exponential (!) modality.

$$\frac{\Gamma \mid \bullet \vdash A \text{ true}}{\Gamma \mid \bullet \vdash {!A} \text{ true}} \; !\, I$$

$$\frac{\Gamma \mid \Delta_1 \vdash {!A} \text{ true} \qquad \Gamma, u : A \text{ valid} \mid \Delta_2 \vdash C \text{ true}}{\Gamma \mid \Delta_1; \Delta_2 \vdash C \text{ true}} \; !\, E$$

A couple of observations. First, the exponential looks quite a bit like the box ($\Box$) modality we learned about in previous lectures. The key difference is that we drop all linear assumptions. Second, we can recover unrestricted behavior by using the exponential around propositions. For example:

$$A \supset B \mapsto {!A} \multimap B$$

### 3.1 Combining Logics

This approach with the exponential is satisfactory from the perspective of provability (we obtain interprovability between logics), but unsatisfactory from the perspective of structure (the translation does not preserve the structure of proofs). Also, practically speaking, if we wanted to combine another logic such as the affine logic, we'd have to tack on yet another sort of context. This means proofs would grow in size and become intractable. A more elegant solution would keep proofs compact. To this end, we introduce the linear non-linear (LNL) logic by Nick Benton [1].

The grammar for LNL is a mix of both structural/unrestricted logic and linear logic, combined with *shifts* of the form $\uparrow$ ("upshift") and $\downarrow$ ("downshift"). We also "tag" the propositions we are considering with the kind of judgment they are subject to. For example, a *linear* proposition would be written $A_L$ true, while an *unrestricted* proposition would be written $A_S$ valid ($S$ is for structural). All the other connectives remain the same.

Grammar:

$$A_S ::= p \mid A_S \wedge B_S \mid A_S \vee B_S \mid A_S \supset B_S \mid A_L \uparrow_L^S$$
$$A_L ::= p \mid A_L \otimes B_L \mid A_L \oplus B_L \mid A_L \multimap B_L \mid A_S \downarrow_L^S$$

In this setting, we call S and L *modes*. So, propositions are either in the structural mode or the linear mode.

Now, how do we go about proving things with this system? First, we consider our judgments. Recall from the lesson on the judgmental S4 that validity cannot depend on truth. Similarly here, proofs of structural propositions cannot depend on linear propositions. Thus, we have two sorts of judgments:

$$\Gamma \mid \Delta \vdash A_L \text{ true}$$
$$\Gamma \mid \cdot \vdash A_S \text{ valid}$$

For each connective, the rules stay largely the same. Of course, we introduced two new connectives in the form of shifts. Their rules go as follows.

$$\frac{\Gamma \mid \cdot \vdash A_L \text{ true}}{\Gamma \mid \Delta \vdash A_L \uparrow_L^S \text{ valid}} \uparrow I \qquad \frac{\Gamma \mid \cdot \vdash A_L \uparrow_L^S \text{ valid}}{\Gamma \mid \cdot \vdash A_L \text{ true}} \uparrow E$$

$$\frac{\Gamma \mid \cdot \vdash A_S \text{ valid}}{\Gamma \mid \cdot \vdash A_S \downarrow_L^S} \downarrow I \qquad \frac{\Gamma \mid \Delta_1 \vdash A_S \downarrow_L^S \qquad \Gamma, A_S \mid \Delta_2 \vdash C_L \text{ true}}{\Gamma \mid \Delta_1; \Delta_2 \vdash C_L \text{ true}} \downarrow E$$

Note that we can model the exponential (!) operator using the upshifts and downshifts.

$$!A \mapsto A_L \uparrow_L^S \downarrow_L^S$$

The whole idea is inspired by semantics – ! is interpreted as a comonad and every comonad comes from an adjunction.

We can do this for other combinations than just linear and structural logics. We will not go into it for these notes, but there is a generalization on LNL called *adjoint logic* that allows us to shift between any number of substructural logics [2].

## References

[1] Benton, P. N. A mixed linear and non-linear logic: Proofs, terms and models. In *International Workshop on Computer Science Logic* (1994), Springer, pp. 121–135.

[2] Jang, J., Roshal, S., Pfenning, F., and Pientka, B. Adjoint natural deduction. In *9th International Conference on Formal Structures for Computation and Deduction (FSCD 2024)* (2024), Schloss Dagstuhl–Leibniz-Zentrum für Informatik, pp. 15–1.