OREGON PROGRAMMING LANGUAGES SUMMER SCHOOL

Modal Logic: Proof Theory, Semantics and Applications

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Lecture 1 - *June 23, 2026*

1 Different Systems of Proofs

Once the syntax and semantics of a logic are fixed, we would like to develop a formal system for deriving its valid formulas. Such formalisms are called proof systems. There are many kinds of proof systems, each with their own style and advantages. Ideally, a proof system should be sound and complete with respect to the semantics: every formula it derives should be semantically valid, and every semantically valid formula should be derivable in the system.

1.1 Hilbert Systems

Hilbert systems are a well-known kind of proof system for deriving formulas. The following Hilbert system is sound and complete for propositional intuitionistic logic. It consists of the following axiom schemata:

K: $\mathcal{A} \to \mathcal{B} \to \mathcal{A}$

S: $(\mathcal{A} \to \mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{B}) \to \mathcal{A} \to \mathcal{C}$

and one inference rule (modus ponens):

$$\frac{A \to B \quad A}{B}\; MP$$

Note: These are called axiom schemas rather than axioms because we instantiate the $\mathcal{A}, \mathcal{B}, \mathcal{C}$s with formulas to produce the axiom instances.

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A derivation in a Hilbert system is a finite sequence of formulas where each element is either an axiom instance or follows from two earlier elements by modus ponens. Here is the sequence for $A \to A$, where A is any formula. Blue, green, and magenta indicate substitutions for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively.

1. $(A \to (B \to A) \to A) \to (A \to (B \to A)) \to A \to A$ (S)
2. $A \to (B \to A) \to A$ (K)
3. $(A \to (B \to A)) \to A \to A$ $(MP1, 2)$
4. $A \to B \to A$ (K)
5. $A \to A$ $(MP3, 4)$

It is challenging to come up with the axiom schema instances to use for the proof sequences. There are more natural ways to write proofs (no pun intended), which are described in the following sections.

1.2 Natural Deduction

natural deduction is the usual way of making inferences in mathematics and it consists of **introduction** and **elimination** rules for each connective.

The following are the inference rules of natural deduction for propositional intuitionistic logic.

- $A \wedge B$

$$\frac{A \quad B}{A \wedge B}\ \wedge I \qquad \frac{A_1 \wedge A_2}{A_i}\ \wedge E_i$$

- $A \vee B$

$$\frac{A_i}{A_1 \vee A_2}\ \vee I \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\ \vee E$$

- $A \to B$

$$\frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \to B} \to I \qquad \frac{A \to B \quad A}{B} \to E$$

- $\bot$

$$\frac{\bot}{C}\ \bot E$$

With these rules, we can build proofs that we call **derivations**.

Example 1.1. *We can prove that $B \wedge C$ is derivable from $A \wedge B$ and C by constructing the following derivation:*

$$\dfrac{\dfrac{A \wedge B}{B}\wedge E_2 \qquad C}{B \wedge C}\wedge I$$

We can also do derivations like this in proof assistants like Lean and Rocq.

Example 1.2. *The proof from the previous example in Lean4 (`https://live.lean-lang.org`) would be:*

```
theorem example : (A ∧ B) → C → B ∧ C := by
  rintro h1 : (A ∧ B)
  rintro h2 : C
  apply And.intro
  . exact And.right h1
  . exact h2
```

1.3 Sequent Calculus

In contrast to Hilbert systems, Gentzen introduced the notion of sequents, which connects two sequences of formulas together by a consequence relation $\vdash$:

$$A_1, \ldots, A_n \vdash B,$$

where we can denote $A_1, \ldots, A_n$ by Γ as the **context** (assumptions).

The following are the rules for the sequent calculus for propositional intuitionistic logic, where every sequent's succedent (the sequence on the right of $\vdash$) contains at most one formula.

- Axiom
$$\dfrac{}{\Gamma, A \vdash A}\text{init}$$

- $A \wedge B$
$$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge R \qquad \dfrac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\wedge L_1 \qquad \dfrac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge L_2$$

- $A \vee B$
$$\dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee R_1 \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee R_2 \qquad \dfrac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\vee L$$

- $A \to B$
$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\to R \qquad \dfrac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\to L$$

- $\bot$

$$\frac{}{\Gamma, \bot \vdash C}\ \bot L$$

Example 1.3. *Prove $B \wedge C$ from $A \wedge B$ and C.*

$$\dfrac{\dfrac{\dfrac{}{B, C \vdash B}\ \text{init}}{A \wedge B, C \vdash B}\ (\wedge L_2) \quad \dfrac{}{A \wedge B, C \vdash C}\ \text{init}}{A \wedge B, C \vdash B \wedge C}\ (\wedge R)$$

An online website to write sequent proofs interactively: `https://seqcalc.dev/`.

1.4 Negation

In intuitionistic logic, we can define negation $\neg A$ as $A \to \bot$ and so, we have two new rules, both in natural deduction

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A}\ \neg I \qquad \dfrac{\neg A \quad A}{\bot}\ \neg E$$

and sequent calculus:

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ \neg R \qquad \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash C}\ \neg L \quad .$$

This, however, is not enough to prove all valid formulas of propositional classical logic. In particular, the excluded middle $A \vee \neg A$ is not provable.

Hence, to switch to the classical setting, we must add one of the equivalents of the excluded middle to natural deduction as an inference rule:

$$\dfrac{\begin{matrix}[\neg A]\\ \vdots \\ \bot\end{matrix}}{A}\ DN$$

and letting the sequent's succedent to have multiple formulas ($\Gamma \vdash \Delta$) to sequent calculus.

Now, we can prove $A \lor \neg A$, as in the following derivation in natural deduction:

$$\dfrac{\dfrac{[\neg(A \lor \neg A)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \lor \neg A)] \quad \dfrac{[A]}{A \lor \neg A}\,\lor_1 I}{\bot}\,\neg E}{\neg A}\,\neg I}{A \lor \neg A}\,\lor_2 I}{\bot}\,\neg E}{A \lor \neg A}\,DN$$

and the following derivation in sequent calculus for propositional classical logic:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,init}{\vdash A, \neg A}\,\neg R}{\vdash A \lor \neg A}\,\lor R$$

This derivation in Lean4 would be:

```
open Classical

theorem example_4 : A ∨ ¬ A := by
  apply byContradiction
  intro (h1 : ¬ (A ∨ ¬ A))
  have h2 : ¬ A := by
    intro (h3 : A)
    have h4 : A ∨ ¬ A := Or.inl h3
    show False
    exact h1 h4
  have h5 : A ∨ ¬ A := Or.inr h2
  show False
  exact h1 h5
```

2 Modal Logic

2.1 Introduction

Classical logic deals with absolute truth. With modalities, we can qualify the truth in various ways. Consider the following variations of the classical sentence 'Carlos is handsome': **Alethic interpretation:** 'Carlos is necessarily/possibly handsome'.

Epistemic interpretation: 'Carlos is known to be handsome'

Doxastic interpretation: 'Carlos is believed to be handsome'

Deontic interpretation: 'Carlos is obligated to be handsome'

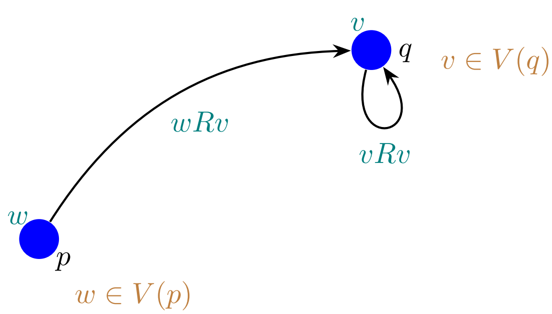
Temporal interpretation: 'Carlos is now/will be handsome'

2.2 (Kripke) Semantics

A relational model is a triple $\mathfrak{M} = \langle W, R, V \rangle$ where

- W is a non-empty set of **possible worlds**.
- $R \subseteq W \times W$ is the relative **accessibility relation**; wRv means from the point of view of w, v is possible.
- $V : At \to \mathcal{P}(W)$ is a valuation function that assigns a truth value to a propositional variable at a world.

Example 2.1. *Here is an example of a simple two-world relational model.*



Formally, the following is the satisfiability relation.

$\mathcal{M}, w \models p$	iff	$w \in V(p)$;
$\mathcal{M}, w \models \bot$		never holds;
$\mathcal{M}, w \models \neg A$	iff	$\mathcal{M}, w \not\models A$;
$\mathcal{M}, w \models A \wedge B$	iff	$\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models A \vee B$	iff	$\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models A \to B$	iff	$\mathcal{M}, w \not\models A$ or $\mathcal{M}, w \models B$;
$\boldsymbol{\mathcal{M}, w \models \Box A}$	**iff**	**for all $\boldsymbol{v}$. $\boldsymbol{wRv}$ implies $\boldsymbol{\mathcal{M}, v \models A}$;**
$\boldsymbol{\mathcal{M}, w \models \Diamond A}$	**iff**	**there exists $\boldsymbol{v}$. $\boldsymbol{wRv}$ and $\boldsymbol{\mathcal{M}, v \models A}$.**

Definition 2.1. *A modal formula A is **satisfiable** if there exists a model $\mathcal{M}$ and $w \in W$ such that $\mathcal{M}, w \vDash A$.*

Definition 2.2 ($\vDash A$)**.** *A modal formula A is **valid** if it is valid in every model.*

Definition 2.3 ($\Gamma \vDash \Delta$)**.** *The argument from a set of formulas Γ to a set of formulas Δ is **valid** if, for every model $\mathcal{M}, w \vDash B$ for each $B \in \Gamma$, then $\mathcal{M}, w \vDash A$ for some $A \in \Delta$.*

Example 2.2. *The relation model from Example 2.1 validates the following formulas.*

$$\mathcal{M}, w \nvDash p \to q \qquad \mathcal{M}, v \models p \to q \qquad \mathcal{M}, w \nvDash \Box p \qquad \mathcal{M}, v \nvDash \Box p$$

$$\mathcal{M}, w \models \Box q \qquad \mathcal{M}, v \models \Box q \qquad \mathcal{M}, w \models \Box(p \to q) \qquad \mathcal{M}, v \models \Box(p \to q)$$

An online tool for constructing relational models via graphical representations: `https://rkirsling.github.io/modallogic/`.