Modal Logic
Part II

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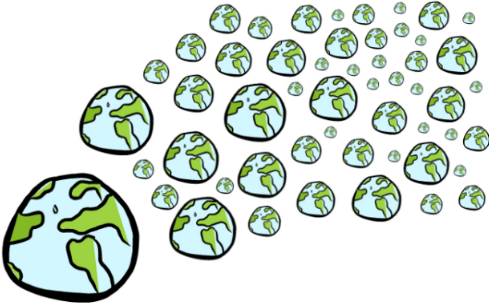


Recap of Lecture I

- Proof theory: structure of proofs and proof systems.
- Proof formalisms:
  - Hilbert systems
  - Natural deduction
  - Sequent calculi
- Brief introduction to modal logic.
- Relational semantics.

In this lecture: more modal logics!

- Dive deep into Kripke semantics.
- Proof theory for modal logics.
- Extensions.

Outline

Outline

Axioms and Kripke

Relating syntax with semantics

Beyond classical and intuitionistic logics

Extensions

Sequent systems

Bonus track: intuitionism

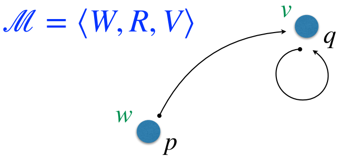
Modal Logics

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$
- **Duality** by De Morgan laws and $\neg\Box A = \Diamond\neg A$
- Axioms: **classical** propositional logic and

$$\mathsf{k}\colon \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \rightarrow B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Relational Models

$\mathcal{M} = \langle W, R, V \rangle$, where $W \neq \emptyset$, $R \subseteq W \times W$ and $V : At \to \wp(W)$.



Recap of Kripke models

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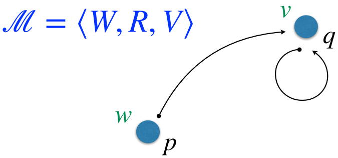
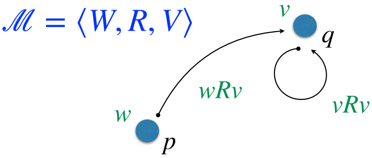


W is a non-empty set of possible worlds.

$\mathcal{M} = \langle W, R, V \rangle$, where $W \neq \emptyset$, $R \subseteq W \times W$ and $V : At \to \wp(W)$.



R is the relative accessibility relation:
from the point of view of w, v is possible.

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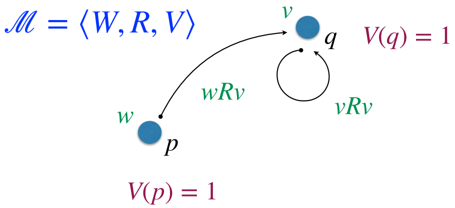

V assigns a truth value to a propositional variable at a world.

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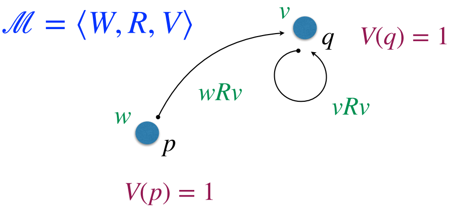

For non-atomic propositional formulas:
Just check the truth table
in each world!

Recap of Kripke models

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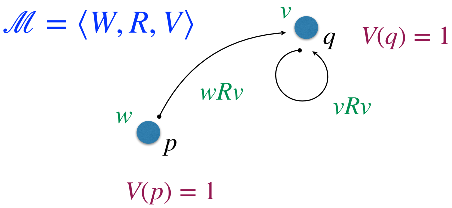

$\mathscr{M}, w \nvDash p \to q$ $\qquad$ $\mathscr{M}, v \vDash p \to q$

Modal Logic Playground

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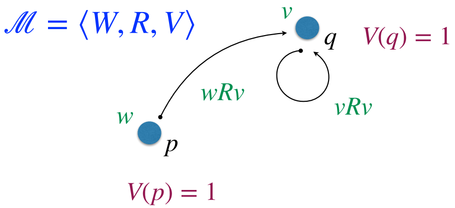

A is *necessary at a world* u provided A is true at every possible world from u.

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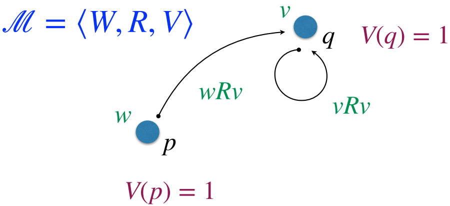

A is *possible at a world* u provided A is true at some possible world from u.

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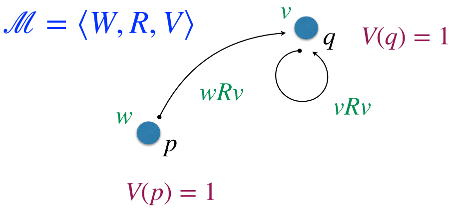

$\mathscr{M}, w \nvDash \Box p$ $\qquad$ $\mathscr{M}, v \nvDash \Box p$

$\mathscr{M}, w \vDash \Box q$ $\qquad$ $\mathscr{M}, v \vDash \Box q$

$\mathscr{M}, w \vDash \Box (p \to q)$ $\qquad$ $\mathscr{M}, v \vDash \Box (p \to q)$

Modal Logic Playground

Recap of Kripke models

$\mathcal{M} = \langle W, R, V \rangle$, where $W \neq \emptyset, R \subseteq W \times W$ and $V : At \to \wp(W)$.

Satisfiability relation $\mathcal{M}, w \models A$:

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & p \in V(w); \\
\mathcal{M}, w \models \bot & & \text{never holds}; \\
\mathcal{M}, w \models \neg A & \text{iff} & \mathcal{M}, w \not\models A; \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models A \vee B & \text{iff} & \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models A \to B & \text{iff} & \mathcal{M}, w \not\models A \text{ or } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models \Box A & \text{iff} & \text{for all } v.\, wRv \text{ implies } \mathcal{M}, v \models A; \\
\mathcal{M}, w \models \Diamond A & \text{iff} & \text{there exists } v.\, wRv \text{ and } \mathcal{M}, v \models A.
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A modal formula A is valid if it is valid in every model ($\models A$).

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\end{array}
$$

A modal formula A is valid if it is valid in every model ($\models A$).

How can I prove that??

😱

The axiom k

$$\mathsf{k}\colon \Box(A \to B) \to (\Box A \to \Box B)$$

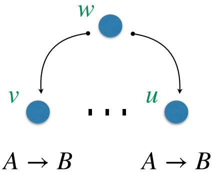
The axiom k

$$\mathsf{k}\colon \Box(A \to B) \to (\Box A \to \Box B)$$

If $\mathcal{M}, w \models \Box(A \to B)$ and $\mathcal{M}, w \models \Box A$ then $\mathcal{M}, w \models \Box B$ for all models $\mathcal{M}$ and all worlds w in it.

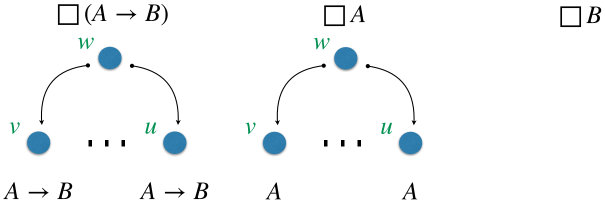
$$\Box(A \to B), \Box A \models \Box B$$

$\Box(A \to B)$ $\Box A$ $\Box B$

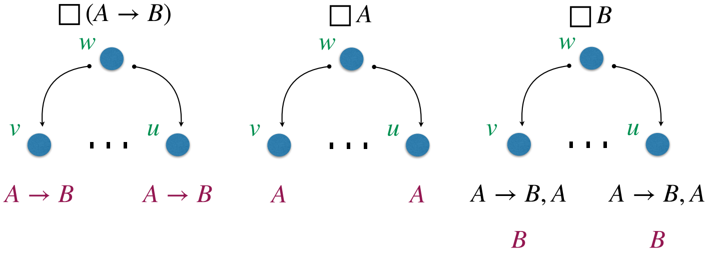


The axiom k

$$\Box(A \to B), \Box A \models \Box B$$

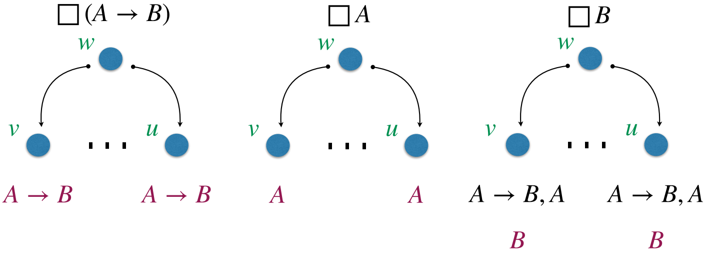


$$\Box(A \to B), \Box A \models \Box B$$



The axiom k

$$\Box(A \to B), \Box A \models \Box B$$



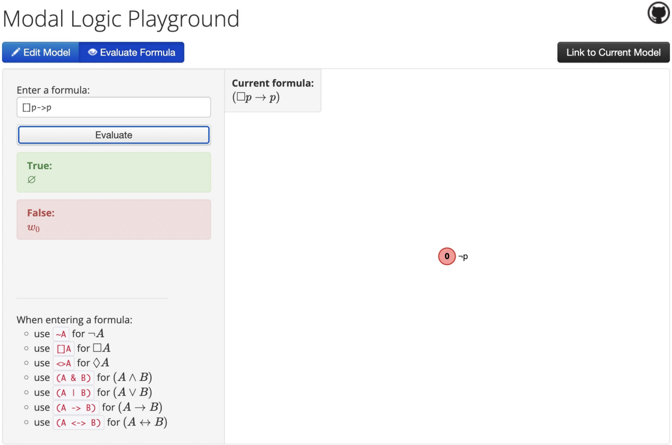
Modal Logic Playground

The axiom T

$$\mathsf{T}\colon \Box A \to A$$

The axiom T

$$\mathsf{T}: \Box A \to A$$

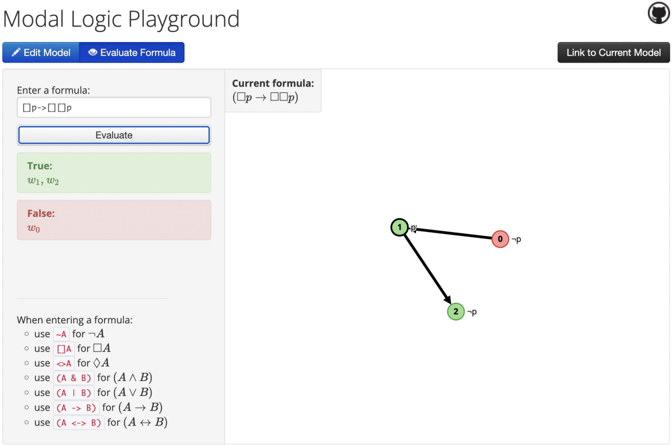


The axiom 4

$$4\colon \Box A \to \Box\Box A$$

The axiom 4

$$4: \Box A \to \Box\Box A$$



Outline

Back to classical logic!

G3cp is the most beautiful system!

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee R \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee L$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \to R \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \to L$$

$$\frac{}{\Gamma, \bot \vdash \Delta} \bot L \qquad \frac{}{\Gamma, A \vdash A, \Delta} \text{ init} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} (\neg R) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} (\neg L)$$

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$$\frac{}{\Gamma, \bot \vdash \Delta} \bot L \qquad \frac{}{\Gamma, A \vdash A, \Delta} \text{init} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} (\neg R) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} (\neg L)$$

- Structural rules are hidden.

Back to classical logic!

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$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

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- Structural rules are hidden.
- All rules are invertible: you may apply any in any order!

Back to classical logic!

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$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee R \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee L$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \rightarrow R \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \rightarrow L$$

$$\frac{}{\Gamma, \bot \vdash \Delta} \bot L \qquad \frac{}{\Gamma, A \vdash A, \Delta} \text{init} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} (\neg R) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} (\neg L)$$

- Structural rules are hidden.
- All rules are invertible: you may apply any in any order!
- In a derivation, if there is one leaf which is not an instance of the initial axiom, then the sequent is not provable!

Back to classical logic!

G3cp is the most beautiful system!

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

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- Structural rules are hidden.
- All rules are invertible: you may apply any in any order!
- In a derivation, if there is one leaf which is not an instance of the initial axiom, then the sequent is not provable!
- Counter-models!

G3cp is the most beautiful system!

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee R \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee L$$

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Example:

$$\frac{}{p \to q, q \vdash p} \to L$$

Back to classical logic!

G3cp is the most beautiful system!

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

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Example:

$$\frac{q \vdash p, p \quad q, q \vdash p}{p \to q, q \vdash p} \to L$$

G3cp is the most beautiful system!

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

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Example:

$$\frac{q \vdash p, p \quad q, q \vdash p}{p \to q, q \vdash p} \to L$$

Counter-model:

$$q \rightsquigarrow T, p \rightsquigarrow F$$

Beyond classical logic?

Question: How about intuitionistic logic?

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G3ip

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee R_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to R \qquad \frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} \to L$$

$$\frac{}{\Gamma, \bot \vdash C} \bot L \qquad \frac{}{\Gamma, A \vdash A} \text{init}$$

Question: How about intuitionistic logic?

G3ip

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L$$

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- Structural rules are hidden.

Question: How about intuitionistic logic?

G3ip

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- Structural rules are hidden.
- $\vee R_1, \vee R_2$ and $\to L$ are not invertible!

Beyond classical logic?

Question: How about intuitionistic logic?

G3ip

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee R_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to R \qquad \frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} \to L$$

$$\frac{}{\Gamma, \bot \vdash C} \bot L \qquad \frac{}{\Gamma, A \vdash A} \text{init}$$

- Structural rules are hidden.
- $\vee R_1, \vee R_2$ and $\to L$ are not invertible!
- Counter-models? (w.r.t. what, btw?)

Question: How about intuitionistic logic?

mLJ

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee R \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B, \Delta} \rightarrow R \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \rightarrow L$$

$$\frac{}{\Gamma, \bot \vdash \Delta} \bot L \qquad \frac{}{\Gamma, A \vdash A, \Delta} \text{init}$$

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$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

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$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge L$$

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- Structural rules are hidden.
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$\mathcal{M} = \langle W, \leq, V \rangle$, where $W \neq \emptyset$ and $V : At \to \wp(W)$.

Relational (Kripke) model for intuitionistic logic

$\mathcal{M} = \langle W, \leq, V \rangle$, where $W \neq \emptyset$ and $V : At \to \wp(W)$.

Satisfiability relation $\mathcal{M}, w \models A$:

$\mathcal{M}, w \models p$	iff	$p \in V(w)$;
$\mathcal{M}, w \models \bot$		never holds;
$\mathcal{M}, w \models \neg A$	iff	$\forall\, w \leq v. \mathcal{M}, v \not\models A$;
$\mathcal{M}, w \models A \wedge B$	iff	$\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models A \vee B$	iff	$\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models A \to B$	iff	for all $v.\, w \leq v$, $\mathcal{M}, v \models A$ implies $\mathcal{M}, v \models B$.

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A formula A is valid if it is valid in every model ($\models A$).

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The argument from a set of formulas Γ to a set of formulas Δ is valid if, for every model $\mathcal{M}$ and every world $w \in W$, if $\mathcal{M}, w \models B$ for each $B \in \Gamma$, then $\mathcal{M}, w \models A$ for some $A \in \Delta$ ($\Gamma \models \Delta$).

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Moreover, truth is monotone w.r.t. propositional variables, that is, $p \in V(w)$ and $w \leq v$ then $p \in V(v)$.

Relational (Kripke) model for intuitionistic logic

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The argument from a set of formulas Γ to a set of formulas Δ is valid if, for every model $\mathcal{M}$ and every world $w \in W$, if $\mathcal{M}, w \models B$ for each $B \in \Gamma$, then $\mathcal{M}, w \models A$ for some $A \in \Delta$ ($\Gamma \models \Delta$).

Theorem 1. (Monotonicity) If $\mathcal{M}, w \models A$ and $w \leq v$ then $\mathcal{M}, v \models A$.

Theorem 2. If $A \to B$ is a tautology then the following rules are admissible in G3cp/G3ip:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta} \qquad\qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Theorem 2. If $A \to B$ is a tautology then the following rules are admissible in G3cp/G3ip:

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Proof. Consider the derivations

$$\dfrac{\dfrac{}{\vdash A \to B} \qquad \dfrac{\Gamma \vdash A, \Delta \qquad \dfrac{}{\Gamma, B \vdash B, \Delta}\,\text{init}}{A \to B, \Gamma \vdash B, \Delta}\,\to L}{\Gamma \vdash B, \Delta}\,\text{cut}$$

and

$$\dfrac{\dfrac{}{\vdash A \to B} \qquad \dfrac{\dfrac{}{\Gamma, A \vdash A, \Delta}\,\text{init} \qquad \Gamma, B \vdash \Delta}{A \to B, \Gamma, A \vdash \Delta}\,\to L}{\Gamma, A \vdash \Delta}\,\text{cut}$$

Transforming semantics into syntax

$$\mathcal{M}, w \models A \wedge B \qquad \text{iff} \qquad \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B$$

Transforming semantics into syntax

$$w : A \wedge B \qquad \leftrightarrow \qquad w : A \wedge w : B$$

Transforming semantics into syntax

$$w : A \wedge B \qquad \leftarrow \qquad w : A \wedge w : B$$

Transforming semantics into syntax

$$w : A \wedge B \qquad \leftarrow \qquad w : A \wedge w : B$$

$$\frac{\Gamma \vdash w : A \wedge w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta}$$

$$w : A \wedge B \qquad \leftarrow \qquad w : A \wedge w : B$$

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta}$$

Transforming semantics into syntax

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$$\frac{\Gamma, w : A \wedge w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta}$$

Transforming semantics into syntax

$$w : A \wedge B \quad \rightarrow \quad w : A \wedge w : B$$

$$\frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta}$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$w : A \wedge w \leq v \quad \rightarrow \quad v : A$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$w : A \wedge w \leq v \quad \rightarrow \quad v : A$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \; \texttt{lift}$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

$$w : A \to B \qquad \leftrightarrow \qquad \forall v.\, w \leq v \wedge v : A \to v : B$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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$$w : A \vee B \qquad \leftrightarrow \qquad w : A \vee w : B$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

$$\frac{\Gamma, w \leq v, v : A \vdash v : B, \Delta}{\Gamma \vdash w : A \to B, \Delta} \to R \qquad \frac{\Gamma \vdash w : A, \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \to B \vdash \Delta} \to L$$

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Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

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Transforming semantics into syntax

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$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma \vdash w : \neg A, \Delta} \neg R \qquad \frac{\Gamma \vdash w : A, \Delta}{\Gamma, w : \neg A \vdash \Delta} \neg L$$

$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

Transforming semantics into syntax

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Let's try to prove $w : p \vee \neg p$

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma \vdash w : \neg A, \Delta} \neg R \qquad \frac{\Gamma \vdash w : A, \Delta}{\Gamma, w : \neg A \vdash \Delta} \neg L$$

$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

Let's try to prove $w : p \vee \neg p$

$$\frac{\overline{}}{\vdash w : p \vee \neg p} \vee R$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma \vdash w : \neg A, \Delta} \neg R \qquad \frac{\Gamma \vdash w : A, \Delta}{\Gamma, w : \neg A \vdash \Delta} \neg L$$

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$$\frac{\overline{\vdash w : p, w : \neg p}}{\vdash w : p \vee \neg p} \vee R$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

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$$\dfrac{\dfrac{wRv, v : p \vdash w : p}{\vdash w : p, w : \neg p} \neg R}{\vdash w : p \vee \neg p} \vee R$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma \vdash w : \neg A, \Delta} \neg R \qquad \frac{\Gamma \vdash w : A, \Delta}{\Gamma, w : \neg A \vdash \Delta} \neg L$$

$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

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$$\dfrac{\dfrac{wRv, v : p \vdash w : p}{\vdash w : p, w : \neg p} \neg R}{\vdash w : p \vee \neg p} \vee R$$

Counter-model:

$$p \rightsquigarrow T \text{ in } v, p \rightsquigarrow F \text{ in } w$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

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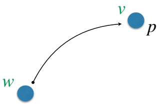
Let's try to prove $w : p \vee \neg p$

$$\dfrac{\dfrac{wRv, v : p \vdash w : p}{\vdash w : p, w : \neg p} \neg R}{\vdash w : p \vee \neg p} \vee R$$

Counter-model:



Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta}\ \wedge R \qquad\qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta}\ \wedge L$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta}\ \texttt{lift}$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma \vdash w : \neg A, \Delta}\ \neg R \qquad\qquad \frac{\Gamma \vdash w : A, \Delta}{\Gamma, w : \neg A \vdash \Delta}\ \neg L$$

$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta}\ \vee R \qquad\qquad \frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta}\ \vee L$$

But wait!

$$\frac{\dfrac{v : \Gamma, w \leq v, v : A \vdash v : B, w : \Delta}{w : \Gamma, w \leq v, v : A \vdash v : B, w : \Delta}\ \texttt{lift}}{w : \Gamma \vdash w : A \to B, w : \Delta}\ {\to} R \quad \rightsquigarrow \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B, \Delta}\ {\to} R$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

Hence

$$\frac{v : \Gamma, A \vdash B}{w : \Gamma \vdash A \to B, \Delta} \to R \qquad \text{in mLJ}$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

Hence

$$\frac{v : \Gamma, A \vdash B}{w : \Gamma \vdash A \rightarrow B} \rightarrow R \qquad \text{in G3ip}$$

Transforming semantics into syntax

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

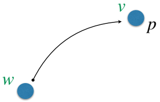
$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma, w \leq v, w : A \vdash \Delta} \texttt{lift}$$

$$\frac{\Gamma, w \leq v, v : A \vdash \Delta}{\Gamma \vdash w : \neg A, \Delta} \neg R \qquad \frac{\Gamma \vdash w : A, \Delta}{\Gamma, w : \neg A \vdash \Delta} \neg L$$

$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

In any case

Counter-model:



Outline

Relational model for modal logic

$\mathcal{M} = \langle W, R, V \rangle$, where $W \neq \emptyset, R \subseteq W \times W$ and $V : At \to \wp(W)$.

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Satisfiability relation $\mathcal{M}, w \models A$:

$\mathcal{M}, w \models p$	iff	$p \in V(w)$;
$\mathcal{M}, w \models \bot$		never holds;
$\mathcal{M}, w \models \neg A$	iff	$\mathcal{M}, w \not\models A$
$\mathcal{M}, w \models A \wedge B$	iff	$\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models A \vee B$	iff	$\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models A \rightarrow B$	iff	$\mathcal{M}, w \not\models A$ or $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models \Box A$	iff	for all $v. wRv$ implies $\mathcal{M}, v \models A$;
$\mathcal{M}, w \models \Diamond A$	iff	there exists $v. wRv$ and $\mathcal{M}, v \models A$.

Relational model for modal logic

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\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & p \in V(w); \\
\mathcal{M}, w \models \bot & & \text{never holds}; \\
\mathcal{M}, w \models \neg A & \text{iff} & \mathcal{M}, w \not\models A \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models A \vee B & \text{iff} & \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models A \to B & \text{iff} & \mathcal{M}, w \not\models A \text{ or } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models \Box A & \text{iff} & \text{for all } v.\, wRv \text{ implies } \mathcal{M}, v \models A; \\
\mathcal{M}, w \models \Diamond A & \text{iff} & \text{there exists } v.\, wRv \text{ and } \mathcal{M}, v \models A.
\end{array}
$$

A modal formula A is valid if it is valid in every model ($\models A$).

The argument from a set of formulas Γ to a set of formulas Δ is valid if, for every model $\mathcal{M}$ and every world $w \in W$, if $\mathcal{M}, w \models B$ for each $B \in \Gamma$, then $\mathcal{M}, w \models A$ for some $A \in \Delta$ ($\Gamma \models \Delta$).

Labeled system: classical modal logic

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w : A \vdash w : B, \Delta}{\Gamma \vdash w : A \rightarrow B, \Delta} \rightarrow R \qquad \frac{\Gamma \vdash w : A, \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \rightarrow B \vdash \Delta} \rightarrow L$$

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$$\frac{\Gamma, wRv \vdash v : A, \Delta}{\Gamma \vdash w : \Box A, \Delta} \Box R \qquad \frac{\Gamma, wRv, v : A \vdash \Delta}{\Gamma, wRv, w : \Box A \vdash \Delta} \Box L$$

Labeled system: classical modal logic

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w : A \vdash w : B, \Delta}{\Gamma \vdash w : A \to B, \Delta} \to R \qquad \frac{\Gamma \vdash w : A, \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \to B \vdash \Delta} \to L$$

$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, v : A \vdash \Delta \quad \Gamma, v : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

$$\frac{\Gamma, wRv \vdash v : A, \Delta}{\Gamma \vdash w : \Box A, \Delta} \Box R \qquad \frac{\Gamma, wRv, v : A \vdash \Delta}{\Gamma, wRv, w : \Box A \vdash \Delta} \Box L$$

Let's try to prove $w : \Box p \to \Box\Box p$

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w : A \vdash w : B, \Delta}{\Gamma \vdash w : A \to B, \Delta} \to R \qquad \frac{\Gamma \vdash w : A, \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \to B \vdash \Delta} \to L$$

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$$\frac{\Gamma, wRv \vdash v : A, \Delta}{\Gamma \vdash w : \Box A, \Delta} \Box R \qquad \frac{\Gamma, wRv, v : A \vdash \Delta}{\Gamma, wRv, w : \Box A \vdash \Delta} \Box L$$

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$$\frac{}{\vdash w : \Box p \to \Box\Box p} \to R$$

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

$$\frac{\Gamma, w : A \vdash w : B, \Delta}{\Gamma \vdash w : A \to B, \Delta} \to R \qquad \frac{\Gamma \vdash w : A, \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \to B \vdash \Delta} \to L$$

$$\frac{\Gamma \vdash w : A, w : B, \Delta}{\Gamma \vdash w : A \vee B, \Delta} \vee R \qquad \frac{\Gamma, v : A \vdash \Delta \quad \Gamma, v : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta} \vee L$$

$$\frac{\Gamma, wRv \vdash v : A, \Delta}{\Gamma \vdash w : \Box A, \Delta} \Box R \qquad \frac{\Gamma, wRv, v : A \vdash \Delta}{\Gamma, wRv, w : \Box A \vdash \Delta} \Box L$$

Let's try to prove $w : \Box p \to \Box\Box p$

$$\dfrac{\dfrac{\overline{}}{w : \Box p \vdash w : \Box\Box p} \Box R}{\vdash w : \Box p \to \Box\Box p} \to R$$

Labeled system: classical modal logic

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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$$\frac{\Gamma, wRv \vdash v : A, \Delta}{\Gamma \vdash w : \Box A, \Delta} \Box R \qquad \frac{\Gamma, wRv, v : A \vdash \Delta}{\Gamma, wRv, w : \Box A \vdash \Delta} \Box L$$

Let's try to prove $w : \Box p \rightarrow \Box\Box p$

$$\cfrac{\cfrac{\cfrac{}{wRv, vRu, w : \Box p \vdash u : p}\ \Box L}{w : \Box p \vdash w : \Box\Box p}\ \Box R}{\vdash w : \Box p \rightarrow \Box\Box p} \rightarrow R$$

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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$$\dfrac{\dfrac{\dfrac{wRv, vRu, v : p \vdash u : p}{wRv, vRu, w : \Box p \vdash u : p} \Box L}{w : \Box p \vdash w : \Box\Box p} \Box R \text{ (double line)}}{\vdash w : \Box p \to \Box\Box p} \to R$$

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$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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Counter-model:

$$p \leadsto T \text{ in } v, p \leadsto F \text{ in } u$$

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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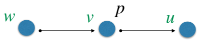

Labeled system: classical modal logic

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta} \wedge R \qquad \frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta} \wedge L$$

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Hence

$$\frac{v : \Gamma \vdash A}{w : \Box\Gamma \vdash \Box A, \Delta} \mathrm{k} \qquad \text{in K}$$

Labeled system: classical modal logic

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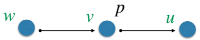
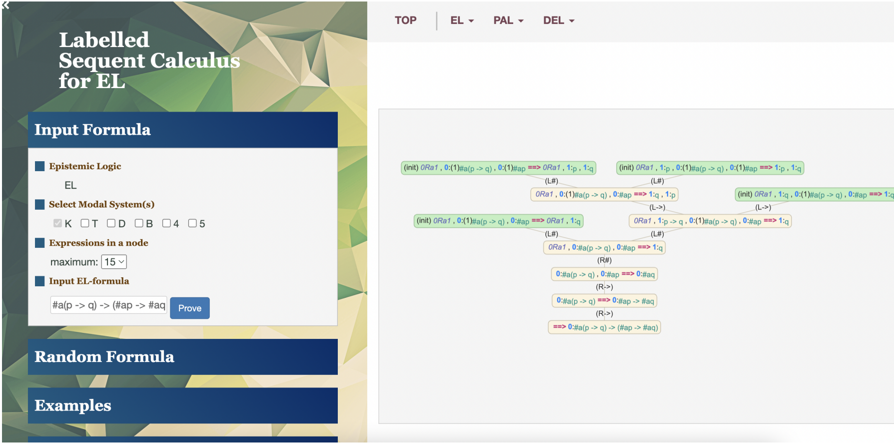
Hence

$$\frac{v : \Gamma \vdash A}{w : \Box\Gamma \vdash \Box A, \Delta} \mathsf{k} \qquad \text{in K}$$

And hence

Counter-model:





Labelled Sequent Calculus for EL
Input Formula
Epistemic Logic
EL
Select Modal System(s)
K T D B 4 5
Expressions in a node
maximum: 15
Input EL-formula
#a(p -> q) -> (#ap -> #aq
Prove
Random Formula
Examples
TOP
EL
PAL
DEL
0:#a(p -> q) , 0:#ap ==> 0:#aq
(R->)
0:#a(p -> q) ==> 0:#ap -> #aq
(R->)
==> 0:#a(p -> q) -> (#ap -> #aq)

ELVis

Outline

Axioms and Kripke

Relating syntax with semantics

Beyond classical and intuitionistic logics

Extensions

Sequent systems

Bonus track: intuitionism

The beautiful correspondence

Principles about modalities $\Longleftrightarrow$ Properties on the accessibility relation

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$\Box A \to A$	$\forall w \,.\, wRw$	Reflexivity

The beautiful correspondence

Principles about modalities $\Longleftrightarrow$ Properties on the accessibility relation

$\Box A \to A$	$\forall w\,.\,wRw$	Reflexivity
$\Box A \to \Box\Box A$	$\forall w, v, u\,.\,wRv \wedge vRu \to wRu$	Transitivity

Principles about modalities ⟺ Properties on the accessibility relation

$\Box A \to A$	$\forall w\,.\,wRw$	Reflexivity
$\Box A \to \Box\Box A$	$\forall w, v, u\,.\,wRv \land vRu \to wRu$	Transitivity
$\Box A \to \Diamond A$	$\forall w\,.\,\exists v\,.\,wRv$	Seriality

The beautiful correspondence

Principles about modalities ⟺ Properties on the accessibility relation

$\Box A \to A$	$\forall w \,.\, wRw$	Reflexivity
$\Box A \to \Box\Box A$	$\forall w, v, u \,.\, wRv \wedge vRu \to wRu$	Transitivity
$\Box A \to \Diamond A$	$\forall w \,.\, \exists v \,.\, wRv$	Seriality
$A \to \Box\Diamond A$	$\forall w, v \,.\, wRv \to vRw$	Symmetry

The beautiful correspondence

Principles about modalities $\Longleftrightarrow$ Properties on the accessibility relation

$\Box A \to A$	$\forall w \,.\, wRw$	Reflexivity
$\Box A \to \Box\Box A$	$\forall w, v, u \,.\, wRv \wedge vRu \to wRu$	Transitivity
$\Box A \to \Diamond A$	$\forall w \,.\, \exists v \,.\, wRv$	Seriality
$A \to \Box\Diamond A$	$\forall w, v \,.\, wRv \to vRw$	Symmetry
$\Diamond A \to \Box\Diamond A$	$\forall w, v, u \,.\, wRv \wedge wRu \to vRu$	Euclideaness

The beautiful modal cube

Axioms ⟺ Modal logics

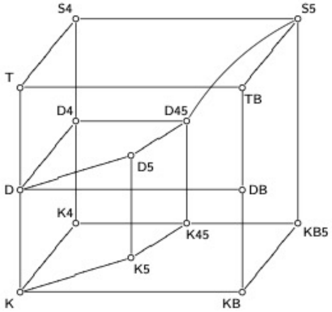
$\mathtt{t}: \quad \Box A \to A$

$\mathtt{4}: \quad \Box A \to \Box\Box A$

$\mathtt{d}: \quad \Box A \to \Diamond A$

$\mathtt{b}: \quad A \to \Box\Diamond A$

$\mathtt{5}: \quad \Diamond A \to \Box\Diamond A$



$\mathtt{k}: \quad \Box(A \to B) \to \Box A \to \Box B$

Axiom T

$T : \Box A \to A$ $\forall w \,.\, wRw$ Reflexivity

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is valid if and only if R is reflexive.

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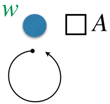

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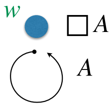

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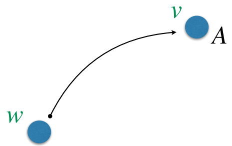

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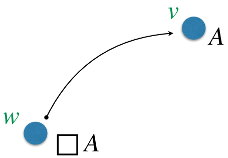

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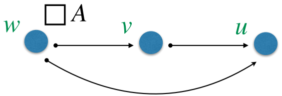

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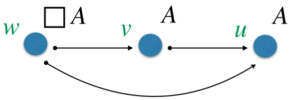
Then, $\forall v, u . wRv$ and vRu, A is valid.



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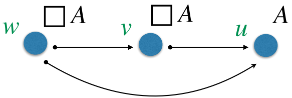

That is, $\mathcal{M}, v \models \Box A$

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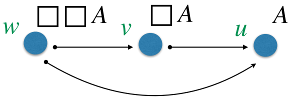

That is, $\mathcal{M}, v \models \Box A$ and hence $\mathcal{M}, w \models \Box\Box A$

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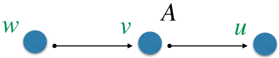

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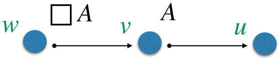

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Outline

$$\mathsf{k} : \Box(A \to B) \to \Box A \to \Box B$$

$$\frac{\Gamma \vdash A}{\Gamma', \Box\Gamma \vdash \Box A, \Delta}\ \mathsf{k}$$

Sequent systems for (normal) modal logics

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$$\frac{\Gamma \vdash A}{\Gamma', \Box\Gamma \vdash \Box A, \Delta}\ \mathsf{k}$$

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$$4 : \Box A \to \Box\Box A$$

$$\frac{\Box\Gamma \vdash A}{\Gamma', \Box\Gamma \vdash \Box A, \Delta}\ 4$$

$$\mathsf{D} : \Box A \to \neg\Box\neg A$$

$$\frac{\Gamma \vdash \bot}{\Gamma', \Box\Gamma \vdash \Delta}\ \mathsf{d}$$

Let S5 $=$ K plus

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Sequent rules (sound and complete):

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\ \mathsf{T} \qquad \frac{\Box\Gamma \vdash A, \Box\Delta}{\Gamma', \Box\Gamma \vdash \Box A, \Box\Delta, \Delta'}\ 45$$

The boundary of sequent calculus

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Not cut-free!!!!!

Theorem 3.
The sequent $p \vdash \Box\Diamond p$ is not cut-free derivable in S5.

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Trivial answer: of course! Take the rules $\{\overline{\vdash A} \mid A \text{ valid in S5}\}$.

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Trivial answer: of course! Take the rules $\{\overline{\vdash A} \mid A \text{ valid in S5}\}$.

Non-trivial answer: it depends on what you call a rule!

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Nested systems, hypersequents, labeled systems.

End of Part II

Thank you!!!

Obrigada!!!

Gracias!!!

☺

Homework II

Implement your own sequent-based theorem prover for modal logics K, KT and S4 in Lean or Rocq.

There will be a companion file with a proposed implementation of propositional classical logic. All you need to do it to extend it to modalities, or just start your own project.

If you want/can, make the prover modular ☺

References and useful links

References:

1. B. F. Chellas, Modal Logic (Cambridge University Press, 1980)
2. Patrick Blackburn, Maarten de Rijke and Yde Venema, Modal Logic (Cambridge University Press, 2001)
3. Sara Negri and Jan von Plato, Structural Proof Theory (Cambridge University Press, 2001)

Useful links:

- L-Framework: `https://carlosolarte.github.io/L-framework/`
- Lean 4 Web: `https://live.lean-lang.org/`
- A Lean intro to Logic: `https://adam.math.hhu.de/#/g/trequetrum/lean4game-logic`
- SeqCalc: `https://seqcalc.dev/`
- ELVis: `https://nomuras.github.io/ELVis/`
- Modal Logic Playground: `https://rkirsling.github.io/modallogic/`
- Accessible theorem provers: `https://staff.cs.manchester.ac.uk/~schmidt/tools/#provers`

You don't need to go classical every time..

File Display Templates Semantics Backward Forward Query Debug Help

```
Reset Initial.
Require Import ProofWeb.
Variables A B: Prop.
Hypothesis P1 : (A -> B).
Hypothesis P2 : (~ B).
Theorem ex_05a : ~A.
Proof.
PBC h1.
neg_e (B).
exact P2.
imp_e A.
exact P1.
PBC h2.
neg_e (~ A).
exact h1.
exact h2.
Qed.
```

```
1 subgoal

  h1 : ~ ~ A
  ============================
   B
```

$$\dfrac{\dfrac{[?]^{P2} \qquad \dfrac{\cdots}{B}}{\bot}\ \neg e}{\neg A}\ \mathrm{PBC}\,[h1]$$

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Theorem ex_05b : ~A.
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neg_i H1.
neg_e (B).
exact P2.
imp_e A.
exact P1.
exact H1.
Qed.
```

```
2 subgoals

  H1 : A
  ============================
   ~ B

subgoal 2 is:
 B
```

$$\dfrac{\dfrac{\dfrac{\dots}{\neg B} \quad \dfrac{\dots}{B}}{\bot}\,\neg e}{\neg A}\,\neg i\,[H1]$$

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- Prove the contrapositive $\neg q \to \neg p$ directly: assume $\neg q$, make intermediary conclusions r_1, r_2 then conclude $\neg p$. Thus, we have also established not only that $\neg q$ implies $\neg p$, but also, that it implies r_1 and r_2 etc. Thus, the proof tells us about what else must be true in worlds where q fails.

... and this may be good for something ☺

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Source: Joel David Hamkins in mathoverflow.

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- Computational problem of trying to find a path in a maze from A to B.

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... and this may be good for something ☺

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Source: Terry Tao in mathoverflow.

Brouwer and intuitionism

FROM FREGE TO GÖDEL

A Source Book in Mathematical Logic, 1879–1931

Jean van Heijenoort

PROFESSOR OF PHILOSOPHY, BRANDEIS UNIVERSITY

HARVARD UNIVERSITY PRESS

CAMBRIDGE, MASSACHUSETTS · 1967

On the significance of the principle of excluded middle in mathematics, especially in function theory

LUITZEN EGBERTUS JAN BROUWER

(*1923b*)

§ 2 shows how several important results of classical analysis become unjustified once the principle of excluded middle is abandoned. Here Brouwer's critique is essentially negative, being based on counterexamples to classical theorems; but elsewhere he investigates which fragments of the Bolzano-Weierstrass theorem can be preserved in intuitionistic analysis (*1919*, sec. 1, and *1952a*; see also *Heyting 1956*, arts. 3.4.4 and 8.1.3) and gives an intuitionistic form of the Heine-Borel theorem (*1926a* and *1926b*; see also *Heyting 1956*, art. 5.2.2).

Brouwer!

The following two fundamental properties, which follow from the principle of excluded middle, have been of basic significance for this incorrect "logical" mathematics of infinity ("logical" because it makes use of the principle of excluded middle), especially for the *theory of real functions* (developed mainly by the Paris school):

1. *The points of the continuum form an ordered point species*;[3]
2. *Every mathematical species is either finite or infinite.*[4]

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The following example shows that the first fundamental property is incorrect. Let d_ν be the νth digit to the right of the decimal point in the decimal expansion of π, and let $m = k_n$ if, as the decimal expansion of π is progressively written, it happens at d_m for the nth time that the segment $d_m d_{m+1} \ldots d_{m+9}$ of this decimal expansion forms the sequence 0123456789. Further, let $c_\nu = (-\frac{1}{2})^{k_1}$ if $\nu \geqq k_1$, otherwise let $c_\nu = (-\frac{1}{2})^\nu$; then the infinite sequence $c_1, c_2, c_3, \ldots$ defines a real number r for which none of the conditions $r = 0$, $r > 0$, or $r < 0$ holds.[5]

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When the first fundamental property ceases to hold, the Paris school's notion of integral, the notion of L-integral, as it is called, ceases to be useful, because this notion of integral is bound to the notion "measurable function" and, according to the above, not even a constant function satisfies the conditions of "measurability". For in the case of the function $f(x) = r$, where r represents the real number defined above, the values of x for which $f(x) > 0$ do not form a measurable point species.[6]

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When the second fundamental property ceases to hold, so does the "extended disjunction principle", according to which, if a fundamental sequence of elements is contained in the union $\mathfrak{S}(p, q)$ of two mathematical species p and q, either p or q contains a fundamental sequence of elements; and when the extended disjunction principle ceases to hold, so does the Bolzano-Weierstrass theorem, which rests upon it and according to which every bounded infinite point species has a limit point.

The following two theorems are less basic and simple than the fundamental properties mentioned, yet they are equally indispensable for the construction of the "logical" theory of functions.

1. *Every continuous function $f(x)$ defined everywhere in a closed interval i possesses a maximum, that is, an abscissa value x_1 having a neighborhood α such that $f(x_1) \geqq f(x)$ for every x that belongs to the intersection of α and i.*

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2. (Heine-Borel covering theorem.) *If a neighborhood is assigned to every point core*[7] *of the point species A formed by the points and the limit points of a bounded entire*[8] *point species B, then the whole point species A can be covered by a finite number of these neighborhoods.*