Type Theory
Solutions to Exercises 1

Rupa's solutions

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Exercise 1

To derive

$$\lambda x{:}\beta \to \alpha.\, \lambda y{:}(\beta \to \alpha) \to \alpha.\, y(\lambda z{:}\beta.\, x\, z) \ : \ (\beta \to \alpha) \to ((\beta \to \alpha) \to \alpha) \to \alpha.$$

Write $\Gamma_0 = x{:}\beta \to \alpha,\ y{:}(\beta \to \alpha) \to \alpha$ and $\Gamma_1 = \Gamma_0,\ z{:}\beta$.

$$\dfrac{\dfrac{\dfrac{\dfrac{y{:}(\beta \to \alpha) \to \alpha \in \Gamma_0}{\Gamma_0 \vdash y : (\beta \to \alpha) \to \alpha} \qquad \dfrac{\dfrac{\dfrac{x{:}\beta \to \alpha \in \Gamma_1}{\Gamma_1 \vdash x : \beta \to \alpha} \qquad \dfrac{z{:}\beta \in \Gamma_1}{\Gamma_1 \vdash z : \beta}}{\Gamma_1 \vdash x\, z : \alpha}}{\Gamma_0 \vdash \lambda z{:}\beta.\, x\, z : \beta \to \alpha}}{\Gamma_0 \vdash y(\lambda z{:}\beta.\, x\, z) : \alpha}}{x{:}\beta \to \alpha \vdash \lambda y{:}(\beta \to \alpha) \to \alpha.\, y(\lambda z{:}\beta.\, x\, z) : ((\beta \to \alpha) \to \alpha) \to \alpha}}{\vdash \lambda x{:}\beta \to \alpha.\, \lambda y{:}(\beta \to \alpha) \to \alpha.\, y(\lambda z{:}\beta.\, x\, z) : (\beta \to \alpha) \to ((\beta \to \alpha) \to \alpha) \to \alpha}$$

Exercise 2

To dress up $\lambda x.\lambda y.\, y(\lambda z.\, x\, z)$ to have type $(\beta \to \gamma) \to ((\beta \to \gamma) \to \alpha) \to \alpha$:

$$\boxed{\lambda x{:}\beta \to \gamma.\, \lambda y{:}(\beta \to \gamma) \to \alpha.\, y(\lambda z{:}\beta.\, x\, z)}$$

Exercise 3

(a)

For $\lambda x{:}C \to E.\, \lambda y{:}(C \to E) \to E.\, y(\lambda z.\, y\, x) : (C \to E) \to ((C \to E) \to E) \to E.$ taking $z{:}C$, giving $\lambda z{:}C.\, y\, x : C \to E$.

Tree form.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[y{:}(C \to E) \to E]^2 \qquad [x{:}C \to E]^1}{y\, x : E}\ \text{(abs } z\text{)}}{\lambda z{:}C.\, y\, x : C \to E} \qquad [y{:}(C \to E) \to E]^2}{y(\lambda z{:}C.\, y\, x) : E}\ (2)}{\lambda y{:}(C \to E) \to E.\, y(\lambda z{:}C.\, y\, x) : ((C \to E) \to E) \to E}\ (1)}{\lambda x{:}C \to E.\, \lambda y{:}(C \to E) \to E.\, y(\lambda z{:}C.\, y\, x) : (C \to E) \to ((C \to E) \to E) \to E}$$

Fitch style.

1	$x : C \to E$	
2	$y : (C \to E) \to E$	
3	$z : C$	
4	$y\,x : E$	(app, 2, 1)
5	$\lambda z{:}C.\, y\,x : C \to E$	(abs, 3, 4)
6	$y(\lambda z{:}C.\, y\,x) : E$	(app, 2, 5)
7	$\lambda y{:}(C \to E) \to E.\, y(\lambda z{:}C.\, y\,x) : ((C \to E) \to E) \to E$	(abs, 2, 6)
8	$\lambda x{:}C \to E.\, \lambda y{:}(C \to E) \to E.\, y(\lambda z{:}C.\, y\,x) : (C \to E) \to ((C \to E) \to E) \to E$	(abs, 1, 7)

(b)

Another term of type $(C \to E) \to ((C \to E) \to E) \to E$:

$$\boxed{\lambda x{:}C \to E.\, \lambda y{:}(C \to E) \to E.\, y\,x}$$

Tree form.

$$\dfrac{\dfrac{\dfrac{[y{:}(C \to E) \to E]^2 \qquad [x{:}C \to E]^1}{y\,x : E}}{\lambda y{:}(C \to E) \to E.\, y\,x : ((C \to E) \to E) \to E}\,(2)}{\lambda x{:}C \to E.\, \lambda y{:}(C \to E) \to E.\, y\,x : (C \to E) \to ((C \to E) \to E) \to E}\,(1)$$

Fitch style.

1	$x : C \to E$	
2	$y : (C \to E) \to E$	
3	$y\,x : E$	(app, 2, 1)
4	$\lambda y{:}(C \to E) \to E.\, y\,x : ((C \to E) \to E) \to E$	(abs, 2, 3)
5	$\lambda x{:}C \to E.\, \lambda y{:}(C \to E) \to E.\, y\,x : (C \to E) \to ((C \to E) \to E) \to E$	(abs, 1, 4)

Exercise 4

The term M should be of the type

$$((\beta \to \gamma) \to \alpha) \to ((\alpha \to \beta) \to \gamma) \to \alpha.$$

1	$x : (\beta \to \gamma) \to \alpha$	[flag]
2	$y : (\alpha \to \beta) \to \gamma$	[flag]
	$?_0 : \alpha$	(goal)

Refining the holes:

$?_0 : \alpha$	use $x : (\beta \to \gamma) \to \alpha$ via app	$\Rightarrow\ ?_0 = x\,?_1, \quad ?_1 : \beta \to \gamma$
$?_1 : \beta \to \gamma$	abstract $z{:}\beta$	$\Rightarrow\ ?_1 = \lambda z{:}\beta.\, ?_2, \quad ?_2 : \gamma$
$?_2 : \gamma$	use $y : (\alpha \to \beta) \to \gamma$ via app	$\Rightarrow\ ?_2 = y\,?_3, \quad ?_3 : \alpha \to \beta$
$?_3 : \alpha \to \beta$	abstract $v{:}\alpha$	$\Rightarrow\ ?_3 = \lambda v{:}\alpha.\, ?_4, \quad ?_4 : \beta$
$?_4 : \beta$	$z : \beta$ is in scope	$\Rightarrow\ ?_4 = z \ \ (v)$

Full Fitch derivation (bottom-up holes filled in).

1	$x : (\beta \to \gamma) \to \alpha$	
	2 $\quad y : (\alpha \to \beta) \to \gamma$	
	3 $\quad z : \beta$	
	4 $\quad v : \alpha$	
	5 $\quad z : \beta$	(in scope
	6 $\quad \lambda v{:}\alpha.\, z : \alpha \to \beta$	(abs, 4, 5
	7 $\quad y(\lambda v{:}\alpha.\, z) : \gamma$	(app, 2, 6
	8 $\quad \lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z) : \beta \to \gamma$	(abs, 3, 7
	9 $\quad x(\lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z)) : \alpha$	(app, 1, 8
	10 $\quad \lambda y{:}(\alpha \to \beta) \to \gamma.\, x(\lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z)) : ((\alpha \to \beta) \to \gamma) \to \alpha$	(abs, 2, 9
11	$\lambda x{:}(\beta \to \gamma) \to \alpha.\, \lambda y{:}(\alpha \to \beta) \to \gamma.\, x(\lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z)) : ((\beta \to \gamma) \to \alpha) \to ((\alpha \to \beta) \to \gamma) \to \alpha$	(abs, 1, 1

Resulting term.

$$\boxed{M := \lambda x{:}(\beta \to \gamma) \to \alpha.\, \lambda y{:}(\alpha \to \beta) \to \gamma.\, x\big(\lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z)\big)}$$

Tree form:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : (\beta \to \gamma) \to \alpha \in \Gamma}{\Gamma \vdash x : (\beta \to \gamma) \to \alpha} \qquad \dfrac{\dfrac{\dfrac{y : (\alpha \to \beta) \to \gamma \in \Gamma}{\Gamma \vdash y : (\alpha \to \beta) \to \gamma} \qquad \dfrac{\dfrac{z : \beta \in \Gamma}{\Gamma \vdash z : \beta}}{\Gamma \vdash \lambda v{:}\alpha.\, z : \alpha \to \beta}}{\Gamma \vdash y(\lambda v{:}\alpha.\, z) : \gamma}}{\Gamma' \vdash \lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z) : \beta \to \gamma}}{\Gamma' \vdash x(\lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z)) : \alpha}}{x{:}(\beta \to \gamma) \to \alpha \vdash \lambda y{:}(\alpha \to \beta) \to \gamma.\, x(\lambda z{:}\beta.\, y(\lambda v{:}\alpha.\, z)) : ((\alpha \to \beta) \to \gamma) \to \alpha}}{\vdash M : ((\beta \to \gamma) \to \alpha) \to ((\alpha \to \beta) \to \gamma) \to \alpha}$$

where $\Gamma' = x{:}(\beta \to \gamma) \to \alpha,\ y{:}(\alpha \to \beta) \to \gamma$.

Exercise 5

Let $A_0 := \alpha$ and $A_{n+1} := A_n \to \alpha$. So

$$A_1 = \alpha \to \alpha, \quad A_2 = (\alpha \to \alpha) \to \alpha, \quad A_3 = ((\alpha \to \alpha) \to \alpha) \to \alpha.$$

(a)

To get a closed term of type $A_3 = ((\alpha \to \alpha) \to \alpha) \to \alpha$
Replace $\lambda x{:}\alpha.\, x : \alpha \to \alpha$ in the argument $f{:}(\alpha \to \alpha) \to \alpha$:

$$\boxed{P_3 := \lambda f{:}(\alpha \to \alpha) \to \alpha.\, f(\lambda x{:}\alpha.\, x)}$$

Exercise 6

(a) Term-constructors for $A + B$

Two injections and one case-elimination:

$$\mathsf{inl} : A \to A + B, \qquad \mathsf{inr} : B \to A + B,$$

$$\mathsf{case} : (A + B) \to (A \to C) \to (B \to C) \to C.$$

Corresponding logical rules ($\vee$):

$$\frac{\Gamma \vdash P : A}{\Gamma \vdash \mathsf{inl}\, P : A + B} \qquad \frac{\Gamma \vdash Q : B}{\Gamma \vdash \mathsf{inr}\, Q : A + B}$$

$$\frac{\Gamma \vdash M : A + B \qquad \Gamma \vdash F : A \to C \qquad \Gamma \vdash G : B \to C}{\Gamma \vdash \mathsf{case}\, M\, F\, G : C}$$

(b) Detour-elimination

A detour is an introduction immediately followed by elimination (case):

$$\boxed{\mathsf{case}\, (\mathsf{inl}\, P)\, F\, G \;\to\; F\, P} \qquad \boxed{\mathsf{case}\, (\mathsf{inr}\, Q)\, F\, G \;\to\; G\, Q}$$