Type Theory
OPLSS 2026
Eugene Oregon

H. Geuvers
Radboud University Nijmegen, NL

Exercises Lecture 2

1. Remember $\bot := \forall\alpha.\alpha$, $\top := \forall\alpha.\alpha{\to}\alpha$
   (a) Verify that in Church $\lambda 2$: $\lambda x{:}\top.x\top x : \top{\to}\top$.
   (b) Verify that in Curry $\lambda 2$: $\lambda x.xx : \top{\to}\top$
   (c) Find a type in Curry $\lambda 2$ for $\lambda x.x\,x\,x$
   (d) Find a type in Curry $\lambda 2$ for $\lambda x.x\,x\,(\lambda y.y\,y)\,x\,x$
   (e) Let $x : \top$. Show that the following term is typable in $\lambda 2$ à la Curry, and give the typing derivation of your result. $\lambda y.\,x\,x\,y\,(\lambda z.y\,y\,z)$.

2. See the slides for definitions.
   (a) Define $\mathrm{inl} : A \to A + B$
   (b) Define the first projection: $\pi_1 : A \times B \to A$
   (c) Define $\mathrm{leaf} : B \to \mathrm{Tree}_{A,B}$ and $\mathrm{join} : \mathrm{Tree}_{A,B} \to \mathrm{Tree}_{A,B} \to A \to \mathrm{Tree}_{A,B}$
   (d) Give the Tree-iteration scheme for $\mathrm{Tree}_{A,B}$ and define $h : \mathrm{Tree}_{A,B} \to \mathrm{Nat}$ that counts the number of leaves of a tree.

3. In $\lambda 2$ à la Church, we have $\mathbb{T}$, the type of ternary trees with leaves in A:

$$\mathbb{T} \quad := \quad \forall\alpha{:}*.(A{\to}\alpha){\to}(\alpha{\to}\alpha{\to}\alpha{\to}\alpha){\to}\alpha.$$

   (a) Define the functions $\mathrm{leaf} : A{\to}\mathbb{T}$ and $\mathrm{join} : \mathbb{T}{\to}\mathbb{T}{\to}\mathbb{T}{\to}\mathbb{T}$.
   (b) Give the function iteration scheme for $\mathbb{T}$. (That is: the scheme for defining functions by iteration on $\mathbb{T}$.)
   (c) Given an element $a_0 : A$, define the function $F : \mathbb{T}{\to}\mathbb{T}$ that replaces, in every node, the middle subtree with a leaf labeled with a_0. (It doesn't change the leaves of the tree.)

4. We let $\mathrm{Str}_A := \exists\alpha.\alpha \times (\alpha \to A) \times (\alpha \to \alpha)$.
   (a) Define $\mathrm{hd} : \mathrm{Str}_A \to A$, taking the head of a stream.
   (b) Define $\mathrm{tl} : \mathrm{Str}_A \to \mathrm{Str}_A$, taking the tail of a stream.
   (c) For B a type with $h : B \to A$ and $t : B \to B$, define $\mathrm{Coit}\,h\,t : B \to \mathrm{Str}_A$, and
   (d) Check that your definitions satisfy the proper equations for a weakly terminal coalgebra:

$$\begin{aligned} \mathrm{hd}\,(\mathrm{Coit}\,h\,t\,x) &=_\beta h\,x \\ \mathrm{tl}\,(\mathrm{Coit}\,h\,t\,x) &=_\beta \mathrm{Coit}\,h\,t\,(t\,x) \end{aligned}$$

   (e) Define the stream of natural numbers, so basically $0, 1, 2, \ldots$ as a term of type $\mathrm{Str}_{\mathrm{Nat}}$.