Type Theory
OPLSS 2026
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Exercises Lecture 4

1. Given the transitive closure of a binary relation, defined in higher order logic style:

$$\begin{aligned}\mathsf{Trclos}\, R \quad := \quad & \lambda x, y{:}A. \\ & (\forall Q{:}A{\to}A{\to}*.(\mathsf{Trans}\, Q \to (R \subseteq Q) \to Q\, x\, y)).\end{aligned}$$

   (a) Write out the types that represent $\mathsf{Trans}\, Q$ and $R \subseteq Q$.

   (b) Prove (by giving a proof term) that the transitive closure is transitive.

   (c) Prove (by giving a proof term) that the transitive closure of R contains R.

2. Given the definition of Leibniz equality

$$t =_A q := \forall P{:}A{\to}*.(Pt \to Pq)$$

   (a) Prove (by giving a proof term) that this equality is reflexive and transitive (easy)

   (b) Prove (by giving a proof term) that this equality is symmetric (Trick: find a "smart" predicate P.)

3. We consider theof the inductively defined equality, using the J-rule. (See the slides, where symmetry is proven.)

   (a) Prove transitivity, that is, give a term

$$\mathsf{trans} : \Pi a, b, c : A.\ a = b \to b = c \to a = c.$$

   (b) Show that, for $p : a = b$, $(p^{-1})^{-1} = \mathsf{refl}$.

4. In the Calculus of Constructions, given $A : *$, $f : A{\to}A$ we define the binary relation J on A as follows

$$J := \lambda x, y{:}A.\, \Pi R{:}A{\to}A{\to}*.(\Pi v, w{:}A.R\, v\, w{\to}R\, w\, v){\to}(\Pi z{:}A.R\, z\, (f\, z)){\to}R\, x\, y.$$

   (a) Give a term of type $\Pi z{:}A.J\, z\, (f\, z)$.

   (b) Prove that J is symmetric, i.e. give a term of type $\Pi x, y{:}A.\ J\, x\, y{\to}J\, y\, x$.

   (c) Give a term of type $\Pi z{:}A.J\, (f\, z)\, z$.