Type Theory

Lecture 3: Dependent Type Theory

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Outline

The idea of dependent types

Dependent Type Theory

Logical Framework

Type Checking Algorithm

Where we are in the course

propositional logic	$\leftrightarrow$	simple type theory $\lambda{\rightarrow}$
2nd order propositional logic	$\leftrightarrow$	polymorphic type theory $\lambda 2$
predicate logic	$\leftrightarrow$	type theory with dependent types λP
higher order predicate logic	$\leftrightarrow$	calculus of constructions λC
univalent foundations	$\leftrightarrow$	homotopy type theory

Extending formulas-as-types to predicate logic

- First order language: domain D, with variables $x, y, z : D$ and possibly functions over D, e.g. $f : D \to D$, $g : D \to D \to D$.
- There are two "kinds" of variables:
  - the first order variables (ranging over the domain D)
  - the "proof variables" (used as [local] assumptions of formulas).
- Formulas and domains are both types. What is the type of a predicate or relation?
- A predicate P is a map from D to the collection of types, $*$
- $P : D \to *$ for P a predicate and $R : D \to D \to *$ for R a binary relation on D.
- We will have to make this more precise ...

Idea of extending to $\forall$

We introduce term rules for the $\forall$-quantifier in predicate logic.

$$\frac{\Gamma \vdash M : \forall x{:}D.A}{\Gamma \vdash M\,t : A[x := t]} \text{ if } t : D \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \lambda x{:}D.M : \forall x{:}D.A}\, x \text{ not free in } \Gamma$$

With the usual β-reduction rule

$$(\lambda x{:}D.M)t \to M[x := t]$$

.

This conforms with cut-elimination (or "detour elimination") on logical derivations.

Example

Deriving irreflexivity from anti-symmetry

$$\begin{aligned}\text{AntiSym}\, R &:= \forall x, y{:}D.(R\,x\,y)\to(R\,y\,x)\to\bot\\ \text{Irrefl}\, R &:= \forall x{:}D.(R\,x\,x)\to\bot\end{aligned}$$

Derivation in predicate logic:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x, y{:}D.R\,x\,y\to R\,y\,x\to\bot}{\forall y{:}D.R\,x\,y\to R\,y\,x\to\bot}}{R\,x\,x\to R\,x\,x\to\bot} \qquad [R\,x\,x]^1}{R\,x\,x\to\bot} \qquad [R\,x\,x]^1}{\bot}}{R\,x\,x\to\bot}\,1}{\forall x{:}D.R\,x\,x\to\bot}$$

Example derivation in type theory, with terms

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{H : \forall x, y{:}D. R\,x\,y \to R\,y\,x \to \bot}{H\,x : \forall y{:}D. R\,x\,y \to R\,y\,x \to \bot}}{H\,x\,x : R\,x\,x \to R\,x\,x \to \bot} \qquad [H' : R\,x\,x]^1}{H\,x\,x\,H' : R\,x\,x \to \bot} \qquad [H' : R\,x\,x]^1}{H\,x\,x\,H'\,H' : \bot}\,1}{\lambda H'{:}(R\,x\,x). H\,x\,x\,H'\,H' : R\,x\,x \to \bot}}{\lambda x{:}A. \lambda H'{:}(R\,x\,x). H\,x\,x\,H'\,H' : \forall x{:}D. R\,x\,x \to \bot}
$$

Main difference between $\lambda\!\to$ and λP

$$A \to B$$

'type of functions from A to B'

$$\Pi x : A.\, B$$

'type of functions from A to B'

dependent product
dependent function type

The type of the function value B now can depend on the function argument x
arrow type becomes a special case.Examples of Π-types:

- $\forall x : \mathbb{N}.x \geq 0$. A proof of $\forall x : \mathbb{N}.x \geq 0$ is a term $p : \Pi x : \mathbb{N}.x \geq 0$, a function that for every $n : \mathbb{N}$ gives a term $p\, n : n \geq 0$.
- $\Pi x : \mathbb{N}.A^x$. A term $t : \Pi x : \mathbb{N}.A^x$ is a function that for every $n : \mathbb{N}$ produces a vectore over A of length n.

Dependent Type Theory

- We have seen informally “dependent types at work” in the predicate logic example.
- Now: the rules

With dependent types:

- everything depends on everything
- we can't first define the types, then the terms
- two universes: $*$ and $\Box$
- $*$ is the universe of types
- We can't have $* : *$, so we have another universe: $* : \Box$.

NB The Coq system uses “Set” and “Prop” for what I call $*$ and “Type” for what I call $\Box$.

First order Dependent Type theory, λP

Derive judgements of the form

$$\Gamma \vdash M : B$$

- Γ is a context

$$x_1 : B_1, x_2 : B_2, \ldots, x_n : B_n$$

- M and B are terms
  taken from the set of pseudoterms

$$\mathsf{T} ::= \mathsf{Var} \mid * \mid \Box \mid (\mathsf{T}\,\mathsf{T}) \mid (\lambda x{:}\mathsf{T}.\mathsf{T}) \mid \Pi x{:}\mathsf{T}.\mathsf{T}$$

Auxiliary judgement

$$\Gamma \vdash$$

denoting that Γ is a correct context.

Derivation rules of λP

$\mathbf{s}$ ranges over $\{*, \Box\}$.

(base) $\emptyset \vdash$ (ctxt) $\dfrac{\Gamma \vdash A : \mathbf{s}}{\Gamma, x{:}A \vdash}$ if x not in Γ (ax) $\dfrac{\Gamma \vdash}{\Gamma \vdash * : \Box}$

(proj) $\dfrac{\Gamma \vdash}{\Gamma \vdash x : A}$ if $x{:}A \in \Gamma$ (Π) $\dfrac{\Gamma \vdash A : * \quad \Gamma, x{:}A \vdash B : \mathbf{s}}{\Gamma \vdash \Pi x{:}A.B : \mathbf{s}}$

(λ) $\dfrac{\Gamma, x{:}A \vdash M : B \quad \Gamma \vdash \Pi x{:}A.B : \mathbf{s}}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B}$ (app) $\dfrac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x := N]}$

(conv) $\dfrac{\Gamma \vdash M : B \quad \Gamma \vdash A : \mathbf{s}}{\Gamma \vdash M : A} A =_\beta B$

Notation: write $A \rightarrow B$ for $\Pi x{:}A.B$ if $x \notin \mathrm{FV}(B)$.

The use of the Π-type

- The Π rule allows to form two forms of function types.

$$(\Pi)\ \frac{\Gamma, x{:}A \vdash B : \mathbf{s} \quad \Gamma \vdash A : *}{\Gamma \vdash \Pi x{:}A.B : \mathbf{s}}$$

$$\Pi x{:}A.B \simeq \{f \mid \forall a : A(f\ a : B[x := a])\}$$

Write $A{\rightarrow}B$ if $x \notin \mathrm{FV}(B)$

- With $\mathbf{s} = *$, we can form $\mathrm{D}{\rightarrow}\mathrm{D}$ and $\Pi x{:}\mathrm{D}.x = x$, etc.
- With $\mathbf{s} = \Box$, we can form $\mathrm{D}{\rightarrow}\mathrm{D}{\rightarrow}*$ and $\mathrm{D}{\rightarrow}*$.

Representation of PRED into λP

PRED = minimal predicate logic, that is: intuitionistic predicate logic with just $\to$ and $\forall$.
Represent both the domains of the logic and the formulas as types.

$$\begin{array}{lcl} \mathsf{A} & : & *, \\ P & : & \mathsf{A}\to *, \\ R & : & \mathsf{A}\to\mathsf{A}\to *, \end{array}$$

Now implication is represented as $\to$ and $\forall$ is represented as Π:

$$\forall x{:}A.P\,x \mapsto \Pi x{:}A.P\,x$$

Intro and elim rules are just λ-abstraction and application

Example

$$\mathsf{A} : *, R : \mathsf{A}\to\mathsf{A}\to* \quad \vdash \quad \lambda z{:}A.\lambda h{:}(\Pi x, y{:}A.R\,x\,y).h\,z\,z$$
$$: \quad \Pi z{:}A.(\Pi x, y{:}A.R\,x\,y)\to R\,z\,z$$

This term is a proof of $\forall z{:}A.(\forall x, y{:}A.R(x, y))\to R(z, z)$

Exercise: Find terms of the following types (NB $\to$ binds strongest)

$$(\Pi x{:}A.P\,x\to Q\,x)\to(\Pi x{:}A.P\,x)\to\Pi x{:}A.Q\,x$$

and

$$(\Pi x{:}A.P\,x\to\Pi z.R\,z\,z)\to(\Pi x{:}A.P\,x)\to\Pi z{:}A.R\,z\,z).$$

Also write down the contexts in which these terms are typed.

Correctness of embedding PRED into λP

For Σ a first order signature, let Γ_Σ denote the associated λP-context.
For φ a PRED-formula, let $\overline{\varphi}$ denote the associated λP-type, and similarly $\overline{\Gamma}$ be a declaration context in λP for a set of formulas Γ.

THEOREM.
For Σ is a first order signature, and Γ a finite set of formulas and φ a formula, we have

$$\Gamma \vdash \varphi \quad \Longleftrightarrow \quad \exists M(\Gamma_\Sigma, \overline{\Gamma} \vdash M : \overline{\varphi})$$

PROOF.
$\Rightarrow$ by induction on the derivation we can construct M as the proof-term that represents the derivation of $\Gamma \vdash \varphi$
$\Leftarrow$ is more complicated. There are terms M that do not "come from a proof" in PRED; this is far from an isomorphism. $\square$

Direct embedding of logic in type theory

For $\lambda\!\rightarrow$ and λP we have seen

Direct representations of logic in type theory.

- Connectives each have a counterpart in the type theory:

implication	$\sim$	$\rightarrow$-type
universal quantification	$\sim$	$\forall$-type

- Logical rules have their direct counterpart in type theory

$\rightarrow$-introduction	$\sim$	λ-abstraction
$\rightarrow$- elimination	$\sim$	application
$\forall$-introduction	$\sim$	λ-abstraction
$\forall$-elimination	$\sim$	application

- Context declares signature, local varibales and assumptions.

LF embedding of logic in type theory

Second way of interpreting logic in type theory De Bruijn; Harper, Honsell, Plotkin:

Logical framework encoding or higher order abstract syntax embedding of logic in type theory.

- Type theory used as a meta system for encoding ones own logic.
- Choose an appropriate context Γ_L, in which the logic L (including its proof rules) is declared.
- Context used as a signature for the logic.
- Use the type system as the 'meta' calculus for dealing with substitution and binding.

Direct and LF embedding

	proof	formula
direct embedding	$\lambda x{:}A.x$	$A \to A$
LF embedding	$\mathrm{imp_intr}\, A\, A\, \lambda x{:}T\, A.x$	$T(A \Rightarrow A)$

- Direct representation:
  One type system : One logic,
  Logical rules $\sim$ type theoretic rules
- LF encoding: One type system : Many logics,
  Logical rules $\sim$ context declarations

Examples of the LF embedding

The encoding of logics in a logical framework is shown by three examples:

1. Minimal proposition logic
2. Minimal predicate logic (just $\{\Rightarrow, \forall\}$)
3. Untyped λ-calculus

Minimal propositional logic

Fix the signature (context) of minimal propositional logic.

$$\begin{array}{rcl} \textbf{prop} & : & * \\ \text{imp} & : & \textbf{prop} \to \textbf{prop} \to \textbf{prop} \end{array}$$

Notation:

$$A \Rightarrow B \text{ for } \text{imp}\, A\, B$$

The type **prop** is the type of 'names' of propositions.
NB : A term of type **prop** can not be inhabited (proved), as it is not a type.
We 'lift' a name p : **prop** to the type of its proofs by declaring the following map:

$$\mathsf{T} \;:\; \textbf{prop} \to * \,.$$

Intended meaning of $\mathsf{T}p$ is 'the type of proofs of p'.
We interpret 'p is valid' by '$\mathsf{T}p$ is inhabited'.

Encoding of derivations

To derive $\mathsf{T}p$ we also encode the logical derivation rules

$$\begin{aligned} \mathsf{imp_intr} &: \Pi p, q : \mathsf{prop}.(\mathsf{T}p \to \mathsf{T}q) \to \mathsf{T}(p \Rightarrow q), \\ \mathsf{imp_el} &: \Pi p, q : \mathsf{prop}.\mathsf{T}(p \Rightarrow q) \to \mathsf{T}p \to \mathsf{T}q. \end{aligned}$$

imp_intr takes two (names of) propositions p and q and a term $f : \mathsf{T}\,p \to \mathsf{T}\,q$ and returns a term of type $\mathsf{T}(p \Rightarrow q)$
Indeed $A \Rightarrow A$, becomes valid:

$$\mathsf{imp_intr} A\, A (\lambda x{:}\mathsf{T}\, A.x) : \mathsf{T}(A \Rightarrow A)$$

Exercise: Construct a term of type $\mathsf{T}(A \Rightarrow (B \Rightarrow A))$

Signature of PROP in LF

To encode proposition logic in LF we need a context (signature) Σ_{PROP}:

$$\begin{array}{rcl} \textbf{prop} & : & * \\ \Rightarrow & : & \textbf{prop} \rightarrow \textbf{prop} \rightarrow \textbf{prop} \\ \mathsf{T} & : & \textbf{prop} \rightarrow * \\ \mathsf{imp_intr} & : & (A, B : \textbf{prop})(\mathsf{T}\, A \rightarrow \mathsf{T}\, B) \rightarrow \mathsf{T}(A \Rightarrow B) \\ \mathsf{imp_el} & : & (A, B : \textbf{prop})\mathsf{T}(A \Rightarrow B) \rightarrow \mathsf{T}\, A \rightarrow \mathsf{T}\, B. \end{array}$$

Desired propertiy of the encoding: Adequacy, in particular

- Soundness of the encoding:

  $$\vdash_{\text{PROP}} A \quad \Longrightarrow \quad \Sigma_{\text{PROP}}, \vec{a}{:}\textbf{prop} \vdash p : \mathsf{T}\, A \text{ for some } p.$$

  $\vec{a}$ is the sequence of proposition variables in A.
- Faithfulness (or completeness) is the converse. It also holds, but more involved to prove.

Minimal predicate logic over one domain A

$$
\begin{array}{rcl}
\textbf{prop} & : & *, \\
\mathsf{A} & : & *, \\
\mathsf{T} & : & \textbf{prop} \rightarrow * \\
\mathsf{f} & : & \mathsf{A} \rightarrow \mathsf{A}, \\
\mathsf{R} & : & \mathsf{A} \rightarrow \mathsf{A} \rightarrow \textbf{prop}, \\
\Rightarrow & : & \textbf{prop} \rightarrow \textbf{prop} \rightarrow \textbf{prop}, \\
\mathsf{imp_intr} & : & \Pi p, q : \textbf{prop}.(\mathsf{T}p \rightarrow \mathsf{T}q) \rightarrow \mathsf{T}(p \Rightarrow q), \\
\mathsf{imp_el} & : & \Pi p, q : \textbf{prop}.\mathsf{T}(p \Rightarrow q) \rightarrow \mathsf{T}p \rightarrow \mathsf{T}q.
\end{array}
$$

Now encode $\forall$:

$\forall$ takes a $P : \mathsf{A} \rightarrow \textbf{prop}$ and returns a proposition, so we add:

$$\forall : (\mathsf{A} \rightarrow \textbf{prop}) \rightarrow \textbf{prop}$$

Universal quantification is translated as follows.

$$\forall x{:}A.(Px) \mapsto \forall(\lambda x{:}A.(Px))$$

Intro and Elim rules for ∀

$$
\begin{array}{rcl}
\forall & : & (\mathrm{A}\to\mathbf{prop})\to\mathbf{prop}, \\
\forall_\mathrm{intr} & : & \Pi P{:}\mathrm{A}\to\mathrm{prop}.(\Pi x{:}\mathrm{A}.\mathrm{T}(Px))\to\mathrm{T}(\forall P), \\
\forall_\mathrm{elim} & : & \Pi P{:}\mathrm{A}\to\mathrm{prop}.\mathrm{T}(\forall P)\to\Pi x{:}\mathrm{A}.\mathrm{T}(Px).
\end{array}
$$

The proof of

$$\forall z{:}A(\forall x, y{:}A.Rxy) \Rightarrow Rzz$$

is now mirrored by the proof-term

$$
\begin{array}{ll}
\forall_\mathrm{intr}[_](& \lambda z{:}\mathrm{A}.\mathrm{imp_intr}[_][_](\lambda h{:}\mathrm{T}(\forall x, y{:}A.Rxy). \\
 & \forall_\mathrm{elim}[_](\forall_\mathrm{elim}[_]hz)z)\ )
\end{array}
$$

We have replaced the instantiations of the Π-type by [_].
This term is of type

$$\mathrm{T}(\forall(\lambda z{:}\mathrm{A}.\mathrm{imp}(\forall(\lambda x{:}\mathrm{A}.(\forall(\lambda y{:}\mathrm{A}.\mathrm{R}xy))))(\mathrm{R}zz)))$$

Intro and Elim rules for $\forall$

$$\begin{aligned}
\forall &: (\mathrm{A}\to\mathbf{prop})\to\mathbf{prop},\\
\forall_\mathrm{intr} &: \Pi P{:}\mathrm{A}\to\mathrm{prop}.(\Pi x{:}\mathrm{A}.\mathrm{T}(P\,x))\to\mathrm{T}(\forall P),\\
\forall_\mathrm{elim} &: \Pi P{:}\mathrm{A}\to\mathrm{prop}.\mathrm{T}(\forall P)\to\Pi x{:}\mathrm{A}.\mathrm{T}(P\,x).
\end{aligned}$$

The proof of

$$\forall z{:}A\,(\forall x,y{:}A.R\,x\,y)\Rightarrow R\,z\,z$$

is now mirrored by the proof-term

$$\begin{aligned}
\forall_\mathrm{intr}[_](\quad & \lambda z{:}\mathrm{A}.\mathrm{imp_intr}[_][_](\lambda h{:}\mathrm{T}(\forall x,y{:}A.Rxy).\\
& \forall_\mathrm{elim}[_](\forall_\mathrm{elim}[_]hz)z)\ )
\end{aligned}$$

Exercise: Construct a proof-term that mirrors the (obvious) proof of

$$\forall x(P\,x\Rightarrow Q\,x)\Rightarrow\forall x.P\,x\Rightarrow\forall x.Q\,x$$

Untyped λ-calculus

$$\text{Signature } \Sigma_{\mathsf{lambda}} : \quad \begin{array}{rcl} \mathsf{D} & : & *; \\ \mathsf{app} & : & \mathsf{D}\to(\mathsf{D}\to\mathsf{D}); \\ \mathsf{abs} & : & (\mathsf{D}\to\mathsf{D})\to\mathsf{D}. \end{array}$$

- A variable x in λ-calculus becomes $x : \mathsf{D}$ in the type system.
- The translation $[-] : \Lambda \to \mathsf{Term}(\mathsf{D})$ is defined as follows.

$$\begin{array}{rcl} [x] & = & x; \\ [PQ] & = & \mathsf{app}\,[P]\,[Q]; \\ [\lambda x.P] & = & \mathsf{abs}\,(\lambda x{:}\mathsf{D}.[P]). \end{array}$$

Examples: $[\lambda x.x\,x] := \mathsf{abs}(\lambda x{:}\mathsf{D}.\mathsf{app}\,x\,x)$
$[(\lambda x.x\,x)\,(\lambda y.y)] := \mathsf{app}\,(\mathsf{abs}\,(\lambda x{:}\mathsf{D}.\mathsf{app}\,x\,x))\,(\mathsf{abs}\,(\lambda y{:}\mathsf{D}.y))$.

Introducing β-equality

$$\mathsf{eq}{:}\mathsf{D}\to\mathsf{D}\to *\,.$$

Notation $P = Q$ for eq P Q.
Rules for proving equalities.

$$\begin{array}{rcl}
\mathsf{refl} & : & \Pi x{:}\mathsf{D}.x = x, \\
\mathsf{sym} & : & \Pi x, y{:}\mathsf{D}.x = y \to y = x, \\
\mathsf{trans} & : & \Pi x, y, z{:}\mathsf{D}.x = y \to y = z \to x = z, \\
\mathsf{mon} & : & \Pi x, x', z, z'{:}\mathsf{D}.x = x' \to z = z' \to \mathsf{app}\ z\ x = \mathsf{app}\ z'\ x', \\
\mathsf{xi} & : & \Pi f, g{:}\mathsf{D}\to\mathsf{D}.(\Pi x{:}\mathsf{D}.f\,x = g\,x) \to \mathsf{abs}\ f = \mathsf{abs}\ g, \\
\mathsf{beta} & : & \Pi f{:}\mathsf{D}\to\mathsf{D}.\Pi x{:}\mathsf{D}.\mathsf{app}\ (\mathsf{abs}\ f)\,x = f\,x.
\end{array}$$

Properties of λP

- Uniqueness of types
  If $\Gamma \vdash M : A$ and $\Gamma \vdash M : B$, then $A =_\beta B$.
- Subject Reduction
  If $\Gamma \vdash M : A$ and $M \rightarrow_\beta N$, then $\Gamma \vdash N : A$.
- Strong Normalization
  If $\Gamma \vdash M : A$, then all β-reductions from M terminate.

Proof of SN is by defining a reduction preserving map from λP to $\lambda\!\rightarrow$.

Decidability Questions

$\Gamma \vdash M : A?$	TCP
$\Gamma \vdash M : ?$	TSP
$\Gamma \vdash ? : A$	TIP

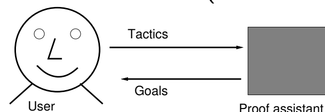
For λP:

- TIP is undecidable
- TCP/TSP: simultaneously with Context checking

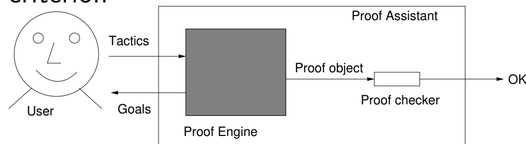
Why would we believe a proof assistant?

Separating the simple proof checker from the powerful proof engine

Proof Assistant (Interactive Theorem Prover)



Proof Assistant with a small kernel that satisfies the De Bruijn criterion



Type Checking (= proof checking)

Define algorithms $\mathrm{Ok}(-)$ and $\mathsf{TC}_{_}(-)$ simultaneously:

- $\mathrm{Ok}(-)$ takes a context and returns 'true' or 'false'
- $\mathsf{TC}_{_}(-)$ takes a context and a term and returns a term or 'false'.

DEFINITION The type synthesis algorithm $\mathsf{TC}_{_}(-)$ is sound if (for all Γ and M)

$$\mathsf{TC}_{\Gamma}(M) = A \implies \Gamma \vdash M : A$$

The type synthesis algorithm $\mathsf{TC}_{_}(-)$ is complete if (for all Γ, M and A)

$$\Gamma \vdash M : A \implies \mathsf{TC}_{\Gamma}(M) =_{\beta} A$$

- A proof assistant like Rocq is based on a type checking algorithm.
- The type checking algorithm is the trusted kernel of Rocq

$$
\begin{array}{rcl}
\mathrm{Ok}(<>) & = & \text{'true'} \\[2ex]
\mathrm{Ok}(\Gamma, x{:}A) & = & \mathsf{TC}_\Gamma(A) \in \{*, \Box\}, \\[2ex]
\mathsf{TC}_\Gamma(x) & = & \text{if } \mathrm{Ok}(\Gamma) \text{ and } x{:}A \in \Gamma \text{ then } A \text{ else 'false'}, \\[2ex]
\mathsf{TC}_\Gamma(*) & = & \text{if } \mathrm{Ok}(\Gamma)\text{then } \Box \text{ else 'false'}, \\[2ex]
\mathsf{TC}_\Gamma(MN) & = & \text{if } \mathsf{TC}_\Gamma(M) = C \text{ and } \mathsf{TC}_\Gamma(N) = D \\
 & & \quad \text{then} \quad \text{if } C \twoheadrightarrow_\beta \Pi x{:}A.B \text{ and } A =_\beta D \\
 & & \quad \qquad\qquad \text{then } B[x := N] \text{ else 'false'} \\
 & & \quad \text{else} \quad\ \text{'false'},
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{TC}_\Gamma(\lambda x{:}A.M) & = & \text{if } \mathsf{TC}_{\Gamma,x:A}(M) = B \\
& & \text{then} \quad \begin{array}{l} \text{if } \mathsf{TC}_\Gamma(\Pi x{:}A.B) \in \{*, \Box\} \\ \text{then } \Pi x{:}A.B \text{ else `false'} \end{array} \\
& & \text{else `false'}, \\
\mathsf{TC}_\Gamma(\Pi x{:}A.B) & = & \text{if } \mathsf{TC}_\Gamma(A) = * \text{ and } \mathsf{TC}_{\Gamma,x:A}(B) = s \\
& & \text{then } s \text{ else `false'}
\end{array}
$$

Soundness and Completeness

Soundness

$$\mathsf{TC}_\Gamma(M) = A \quad\Longrightarrow\quad \Gamma \vdash M : A$$

Completeness

$$\Gamma \vdash M : A \quad\Longrightarrow\quad \mathsf{TC}_\Gamma(M) =_\beta A$$

As a consequence:

$$\mathsf{TC}_\Gamma(M) = \text{‘false’} \quad\Longrightarrow\quad M \text{ is not typable in } \Gamma$$

NB 1. Completeness implies that TC terminates on all well-typed terms. We want that TC terminates on all pseudo terms.
NB 2. Completeness only makes sense if we have uniqueness of types
(Otherwise: let $\mathsf{TC}_{_}(-)$ generate a set of possible types)

Termination

We want $\mathsf{TC}_{_}(-)$ to terminate on all inputs.
Interesting cases: λ-abstraction and application:

$$\begin{array}{rcl}\mathsf{TC}_\Gamma(\lambda x{:}A.M) & = & \text{if } \mathsf{TC}_{\Gamma,x:A}(M) = B \\ & & \quad \text{then} \qquad \text{if } \mathsf{TC}_\Gamma(\Pi x{:}A.B) \in \{*, \Box\} \\ & & \qquad\qquad\qquad \text{then } \Pi x{:}A.B \text{ else `false'} \\ & & \quad \text{else `false'},\end{array}$$

! Recursive call is not on a smaller term!
Replace the side condition

$$\text{if } \mathsf{TC}_\Gamma(\Pi x{:}A.B) \in \{*, \Box\}$$

by

$$\text{if } \mathsf{TC}_\Gamma(A) \in \{*\}$$

Termination

We want $\mathsf{TC}_{_}(-)$ to terminate on all inputs.
Interesting cases: λ-abstraction and application:

$$
\begin{array}{rcl}
\mathsf{TC}_\Gamma(MN) & = & \text{if } \mathsf{TC}_\Gamma(M) = C \text{ and } \mathsf{TC}_\Gamma(N) = D \\
 & & \quad \text{then} \quad \text{if } C \twoheadrightarrow_\beta \Pi x{:}A.B \text{ and } A =_\beta D \\
 & & \quad \phantom{\text{then}} \quad \text{then } B[x := N] \text{ else `false'} \\
 & & \quad \text{else} \quad \text{`false'},
\end{array}
$$

! Need to decide β-reduction and β-equality!
For this case, termination follows from:

- Soundness of TC and
- Decidability of equality on well-typed terms.

This decidability of equality follows from SN (strong normalization) and CR (Church-Rosser property).

Questions?